# Bilinear structure and KP reductions to a coupled Sasa-Satsuma equation


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# Contents



This is a joint work with my advisor Dr. Baofeng Feng, Dr. Chengfa Wu at Shenzhen University and his student Guangxiong Zhang.

# Sasa-Satsuma equation

1. The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\frac{\partial q}{\partial T} + \frac{1}{2}\frac{\partial^2 q}{\partial X^2} + |q|^2 q = 0, \quad (1)$$

arises as a generic model in water waves, nonlnear optics, plasma and many other fileds.

2. The Sasa-Satsuma equation

$$\mathrm{i}\frac{\partial q}{\partial T} + \frac{1}{2}\frac{\partial^2 q}{\partial X^2} + |q|^2 q + \mathrm{i}\epsilon\left\{\frac{\partial^3 q}{\partial X^3} + 6|q|^2\frac{\partial q}{\partial X} + 3q\frac{\partial |q|^2}{\partial X}\right\} = 0, \quad (2)$$

is an integrable extension to the above NLS equation, which can be converted into

$$u_t = u_{xxx} + 6|u|^2 u_x + 3u\left(|u|^2\right)_x. \quad (3)$$

by taking $t = -T\epsilon$, $x = X - T/(12\epsilon)$ and
$u(x,t) = q(X,T)\exp\left\{-\frac{\mathrm{i}}{6\epsilon}\left(X - \frac{T}{18\epsilon}\right)\right\}$.

# Coupled Sasa-Satsuma equations

The Sasa-Satsuma equation has several integrable multi-component extensions

- CSSI equation

$$u_{1t} + u_{1xxx} - 6c\left(|u_1|^2 + |u_2|^2\right)u_{1x} - 3c\left(|u_1|^2 + |u_2|^2\right)_x u_1 = 0, \quad \text{(4a)}$$

$$u_{2t} + u_{2xxx} - 6c\left(|u_1|^2 + |u_2|^2\right)u_{2x} - 3c\left(|u_1|^2 + |u_2|^2\right)_x u_2 = 0. \quad \text{(4b)}$$

  Above equation degenerates to (3) under $u_2 = u_1$.

- CSSII equation

$$u_{1t} + u_{1xxx} - 6c\left(|u_1|^2 + |u_2|^2\right)u_{1x} - 6c\left(u_1^* u_{1x} + u_2^* u_{2x}\right)u_1 = 0, \quad \text{(5a)}$$

$$u_{2t} + u_{2xxx} - 6c\left(|u_1|^2 + |u_2|^2\right)u_{2x} - 6c\left(u_1^* u_{1x} + u_2^* u_{2x}\right)u_2 = 0, \quad \text{(5b)}$$

  it degenerates to (3) under $u_2 = u_1^*$.

We are going to present results about CSSI equation (4a)-(4b) in this talk.

# Bilinearization of CSSI equation under nonzero b.c. (I)

**Lemma 1.**

*The coupled Sasa-Satsuma equation* (4a)-(4b) *can be transformed into a set of 6 bilinear equations*

$$\begin{aligned}
&(D_x+2i\alpha_1)g_1\cdot g_1^* = 2i\alpha_1 q_{11} f\\
&(D_x+2i\alpha_2)g_2\cdot g_2^* = 2i\alpha_2 q_{22} f\\
&(D_x+i\alpha_1-i\alpha_2)g_1\cdot g_2 = (i\alpha_1-i\alpha_2)s_{12} f\\
&(D_x^2-8c)f\cdot f+4cg_1g_1^*+4cg_2g_2^*=0\\
&(D_x^3-D_t+3i\alpha_1D_x^2-3(\alpha_1^2+8c)D_x-12ic\alpha_1)g_1\cdot f+6c[i\alpha_1q_{11}g_1+i\alpha_2q_{22}g_1+i(\alpha_1-\alpha_2)s_{12}g_2^*]=0\\
&(D_x^3-D_t+3i\alpha_2D_x^2-3(\alpha_2^2+8c)D_x-12ic\alpha_2)g_2\cdot f+6c[i\alpha_1q_{11}g_2+i\alpha_2q_{22}g_2+i(\alpha_2-\alpha_1)s_{12}g_1^*]=0
\end{aligned} \tag{6}$$

*via the variable transformation*

$$u_1=\frac{g_1}{f}e^{i(\alpha_1(x-12ct)-\alpha_1^3t)},\quad u_2=\frac{g_2}{f}e^{i(\alpha_2(x-12ct)-\alpha_2^3t)} \tag{7}$$

The Hirota's bilinear operator

$$D_x^mD_t^nf\cdot g=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n\left[f(x,t)g\left(x',t'\right)\right]\Big|_{x'=x,t'=t} \tag{8}$$

Substitution into the first equation:

$$
\begin{aligned}
& f^2(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2 + 8c)D_x + 12\mathrm{i}c\alpha_1)g_1 \cdot f \\
& -3(D_x g_1 \cdot f)[(D_x^2 - 8c)f \cdot f + 4cg_1 g_1^* + 4cg_2 g_2^*] \\
+ \quad & 3cg_1 f[(D_x - 2\mathrm{i}\alpha_1)g_1 \cdot g_1^* + (D_x - 2\mathrm{i}\alpha_1)g_2 \cdot g_2^*] \\
& -3\mathrm{i}\alpha_1 g_1 f D_x^2 f \cdot f + 6cg_2^* f D_x g_1 \cdot g_2 = 0.
\end{aligned}
$$

By requiring

$$
(D_x^2 - 8c)f \cdot f + 4cg_1 g_1^* + 4cg_2 g_2^* = 0,
$$

We have

$$
\begin{aligned}
& f^2(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2 + 8c)D_x - 12\mathrm{i}c\alpha_1)g_1 \cdot f \\
& +3cg_1 f(D_x + 2\mathrm{i}\alpha_1)g_1 \cdot g_1^* \\
+ \quad & 3cg_1 f[(D_x + 2\mathrm{i}\alpha_1)g_2 \cdot g_2^*] + 6cg_2^* f D_x g_1 \cdot g_2 = 0,
\end{aligned}
$$

# Bilinearization of CSSI equation under nonzero b.c. (II)

Substitution into the first equation:

$$\begin{aligned} & f^2(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2+8c)D_x + 12\mathrm{i}c\alpha_1)g_1\cdot f \\ & -3(D_x g_1\cdot f)[(D_x^2-8c)f\cdot f + 4cg_1g_1^* + 4cg_2g_2^*] \\ + \quad & 3cg_1 f[(D_x - 2\mathrm{i}\alpha_1)g_1\cdot g_1^* + (D_x-2\mathrm{i}\alpha_1)g_2\cdot g_2^*] \\ & -3\mathrm{i}\alpha_1 g_1 f D_x^2 f\cdot f + 6cg_2^* f D_x g_1\cdot g_2 = 0. \end{aligned}$$

By requiring

$$(D_x^2-8c)f\cdot f + 4cg_1g_1^* + 4cg_2g_2^* = 0,$$

We have

$$\begin{aligned} & f^2(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2+8c)D_x - 12\mathrm{i}c\alpha_1)g_1\cdot f \\ & +3cg_1 f(D_x+2\mathrm{i}\alpha_1)g_1\cdot g_1^* \\ + \quad & 3cg_1 f[(D_x+2\mathrm{i}\alpha_1)g_2\cdot g_2^*] + 6cg_2^* f D_x g_1\cdot g_2 = 0, \end{aligned}$$

$$\begin{aligned} & f^2(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2+8c)D_x - 12\mathrm{i}c\alpha_1)g_1\cdot f \\ & +3cg_1 f(D_x+2\mathrm{i}\alpha_1)g_1\cdot g_1^* \\ + \quad & 3cg_1 f[(D_x+2\mathrm{i}\alpha_2)g_2\cdot g_2^*] + 6cg_2^* f[(D_x + \mathrm{i}\alpha_1 - \mathrm{i}\alpha_2)g_1\cdot g_2] = 0, \end{aligned}$$

# Bilinearization of CSSI equation under nonzero b.c. (III)

Introducing $q_{11}, q_{22}$ and $s_{12}$ as auxiliary functions, we have

$$\begin{aligned}
&(D_x + \mathrm{i}\alpha_1 - \mathrm{i}\alpha_2) g_1 \cdot g_2 = (\mathrm{i}\alpha_1 - \mathrm{i}\alpha_2) s_{12} f, \\
&(D_x + 2\mathrm{i}\alpha_1) g_1 \cdot g_1^* = 2\mathrm{i}\alpha_1 q_{11} f, \\
&(D_x + 2\mathrm{i}\alpha_2) g_2 \cdot g_2^* = 2\mathrm{i}\alpha_2 q_{22} f
\end{aligned}$$

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&(D_x + 2\mathrm{i}\alpha_1) g_1 \cdot g_1^* = 2\mathrm{i}\alpha_1 q_{11} f, \\
&(D_x + 2\mathrm{i}\alpha_2) g_2 \cdot g_2^* = 2\mathrm{i}\alpha_2 q_{22} f
\end{aligned}$$

$$\begin{aligned}
&(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2 + 8c) D_x - 12\mathrm{i}c\alpha_1) g_1 \cdot f \\
&+6c[\mathrm{i}\alpha_1 q_{11} g_1 + \mathrm{i}\alpha_2 q_{22} g_1 + \mathrm{i}(\alpha_1 - \alpha_2) s_{12} g_2^*] = 0,
\end{aligned}$$

$$(D_x^2 - 8c) f \cdot f + 4c g_1 g_1^* + 4c g_2 g_2^* = 0$$

# Bilinearization of CSSI equation under nonzero b.c. (III)

Introducing $q_{11}, q_{22}$ and $s_{12}$ as auxiliary functions, we have

$$\begin{aligned}&(D_x + \mathrm{i}\alpha_1 - \mathrm{i}\alpha_2)g_1 \cdot g_2 = (\mathrm{i}\alpha_1 - \mathrm{i}\alpha_2)s_{12}f,\\ &(D_x + 2\mathrm{i}\alpha_1)g_1 \cdot g_1^* = 2\mathrm{i}\alpha_1 q_{11}f,\\ &(D_x + 2\mathrm{i}\alpha_2)g_2 \cdot g_2^* = 2\mathrm{i}\alpha_2 q_{22}f\end{aligned}$$

$$\begin{aligned}&(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2 + 8c)D_x - 12\mathrm{i}c\alpha_1)g_1 \cdot f\\ &+6c[\mathrm{i}\alpha_1 q_{11}g_1 + \mathrm{i}\alpha_2 q_{22}g_1 + \mathrm{i}(\alpha_1 - \alpha_2)s_{12}g_2^*] = 0,\end{aligned}$$

$$(D_x^2 - 8c)f \cdot f + 4cg_1g_1^* + 4cg_2g_2^* = 0$$

From the second equation

$$\begin{aligned}&(D_x^3 - D_t + 3\mathrm{i}\alpha_2 D_x^2 - 3(\alpha_2^2 + 8c)D_x - 12\mathrm{i}c\alpha_2)g_2 \cdot f\\ &+ 6c[\mathrm{i}\alpha_1 q_{11}g_2 + \mathrm{i}\alpha_2 q_{22}g_2 + \mathrm{i}(\alpha_2 - \alpha_1)s_{12}g_1^*] = 0\end{aligned}$$

Define the $\tau_{k_1,l_1,k_2,l_2}$ function by

$$\tau_{k_1,l_1,k_2,l_2} = \det\left(m_{ij}^{k_1,l_1,k_2,l_2}\right)_{1\le i,j\le N}$$

where the matrix element is defined as

$$m_{ij}^{k_1,l_1,k_2,l_2} = c_{ij} + \sum_{m,n=1}^{K}\left[\frac{1}{p_{im}+\bar{p}_{jn}}\left(-\frac{p_{im}-a_1}{\bar{p}_{jn}+a_1}\right)^{k_1}\left(-\frac{p_{im}-b_1}{\bar{p}_{jn}+b_1}\right)^{l_1}\right.$$
$$\left.\times\left(-\frac{p_{im}-a_2}{\bar{p}_{jn}+a_2}\right)^{k_2}\left(-\frac{p_{im}-b_2}{\bar{p}_{jn}+b_2}\right)^{l_2} e^{\xi_{im}+\bar{\xi}_{jn}}\right],$$

$$\xi_{im} = p_{im}x + p_{im}^2 y + p_{im}^3 t + \frac{1}{p_{im}-a_1}r_1 + \frac{1}{p_{im}-b_1}s_1$$
$$+\frac{1}{p_{im}-a_2}r_2 + \frac{1}{p_{im}-b_2}s_2 + \xi_{im,0},$$

$$\bar{\xi}_{jn} = \bar{p}_{jn}x - \bar{p}_{jn}^2 y + \bar{p}_{jn}^3 t + \frac{1}{\bar{p}_{jn}+a_1}r_1 + \frac{1}{\bar{p}_{jn}+b_1}s_1$$
$$+\frac{1}{\bar{p}_{jn}+a_2}r_2 + \frac{1}{\bar{p}_{jn}+b_2}s_2 + \bar{\xi}_{jn,0}$$

Above $\tau$ functions satisfy the following bilinear equations from the KP-Toda Hierarchy

$$\left(D_{r_1}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1+1,l_1,k_2,l_2}\tau_{k_1-1,l_1,k_2,l_2} \tag{9}$$

$$\left(D_{s_1}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1,l_1+1,k_2,l_2}\tau_{k_1,l_1-1,k_2,l_2} \tag{10}$$

$$\left(D_{r_2}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2-1,l_2} \tag{11}$$

$$\left(D_{s_2}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1,l_1,k_2,l_2+1}\tau_{k_1,l_1,k_2,l_2-1} \tag{12}$$

$$(D_x + a_1 - b_1)\,\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1+1,k_2,l_2} = (a_1 - b_1)\tau_{k_1+1,l_1+1,k_2,l_2}\tau_{k_1,l_1,k_2,l_2} \tag{13}$$

$$(D_x + a_2 - b_2)\,\tau_{k_1,l_1,k_2+1,l_2}\cdot\tau_{k_1,l_1,k_2,l_2+1} = (a_2 - b_2)\tau_{k_1,l_1,k_2+1,l_2+1}\tau_{k_1,l_1,k_2,l_2} \tag{14}$$

$$(D_x + a_1 - a_2)\,\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2+1,l_2} = (a_1 - a_2)\tau_{k_1+1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2,l_2} \tag{15}$$

$$\left(D_x^2 - D_y + 2a_1D_x\right)\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = 0 \tag{16}$$

$$\left(D_x^3 + 3D_xD_y - 4D_t + 3a_1\left(D_x^2 + D_y\right) + 6a_1^2D_x\right)\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = 0 \tag{17}$$

$$\left(D_{r_1}\left(D_x^2 - D_y + 2a_1D_x\right) - 4D_x\right)\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = 0 \tag{18}$$

$$\begin{aligned}&\left(D_{s_1}\left(D_x^2 - D_y + 2a_1D_x\right) - 4\left(D_x + a_1 - b_1\right)\right)\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2}\\&\quad+4(a_1 - b_1)\tau_{k_1+1,l_1+1,k_2,l_2}\cdot\tau_{k_1,l_1-1,k_2,l_2} = 0\end{aligned} \tag{19}$$

$$\begin{aligned}&\left(D_{r_2}\left(D_x^2 - D_y + 2a_1D_x\right) - 4\left(D_x + a_1 - a_2\right)\right)\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2}\\&\quad+4(a_1 - a_2)\tau_{k_1+1,l_1,k_2+1,l_2}\cdot\tau_{k_1,l_1,k_2-1,l_2} = 0\end{aligned} \tag{20}$$

$$\begin{aligned}&\left(D_{s_2}\left(D_x^2 - D_y + 2a_1D_x\right) - 4\left(D_x + a_1 - b_2\right)\right)\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2}\\&\quad+4(a_1 - b_2)\tau_{k_1+1,l_1,k_2,l_2+1}\cdot\tau_{k_1,l_1,k_2,l_2-1} = 0\end{aligned} \tag{21}$$

$$\begin{aligned}&(b_2 - a_1)\tau_{k_1+1,l_1,k_2,l_2+1}\tau_{k_1,l_1,k_2+1,l_2} + (a_2 - b_2)\tau_{k_1,l_1,k_2+1,l_2+1}\tau_{k_1+1,l_1,k_2,l_2}\\&\quad+(a_1 - a_2)\tau_{k_1+1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2,l_2+1} = 0\end{aligned} \tag{22}$$

# AKP $\to$ CKP reduction

We start with the reduction from AKP to CKP. To realize this, we let $N = 2M$ and impose the parameter constraints

$$\bar{p}_{jn} = p_{jn}, \quad b_n = -a_n, \quad \xi_{jn,0} = \bar{\xi}_{jn,0}, \quad j = 1, 2, \ldots, N, \quad n = 1, 2,$$

then we can prove that

$$\tau_{k_1,l_1,k_2,l_2}(x, y, t, r_1, s_1, r_2, s_2) = \tau_{-l_1,-k_1,-l_2,-k_2}(x, -y, t, s_1, r_1, s_2, r_2).$$

# The complex conjugate reduction

Let $a_1 = \mathrm{i}\alpha_1$ and $a_2 = \mathrm{i}\alpha_2$ be purely imaginary. Further, by imposing the parameter relations

$$p_{2M+1-j} = p_j^*, \quad \xi_{2M+1-j,0} = \xi_{j,0}^*$$

we obtain

$$\tau^*_{0,k_1,0,k_2} = \tau_{k_1,0,k_2,0}, \qquad \tau^*_{k_1,k_1,k_2,k_2} = \tau_{k_1,k_1,k_2,k_2},$$
$$\tau^*_{0,k_1,k_2,k_2} = \tau_{k_1,0,k_2,k_2}, \quad \tau^*_{k_1,k_1,0,k_2} = \tau_{k_1,k_1,k_2,0},$$
$$\tau^*_{0,k_1,k_2,0,} = \tau_{k_1,0,0,k_2}.$$

This indicates that $\tau_{k_1,k_1,k_2,k_2}$ is real. Define

$$f = \tau_{0000}, \quad g_1 = \tau_{1000}, \quad g_2 = \tau_{0010}, \quad h_1 = \tau_{0100} = g_1^*$$
$$h_2 = \tau_{0001} = g_2^*, \quad q_{11} = \tau_{1100}, \quad q_{22} = \tau_{0011}, \quad s_{12} = \tau_{1010},$$

# KP-Toda Reduction (I)

$$(D_x + a_1 - b_1)\,\tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1+1,k_2,l_2} = (a_1 - b_1)\tau_{k_1+1,l_1+1,k_2,l_2}\tau_{k_1,l_1,k_2,l_2}$$
$$(D_x + a_2 - b_2)\,\tau_{k_1,l_1,k_2+1,l_2} \cdot \tau_{k_1,l_1,k_2,l_2+1} = (a_2 - b_2)\tau_{k_1,l_1,k_2+1,l_2+1}\tau_{k_1,l_1,k_2,l_2}$$
$$(D_x + a_1 - a_2)\,\tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1,k_2+1,l_2} = (a_1 - a_2)\tau_{k_1+1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2,l_2}$$

# KP-Toda Reduction (I)

$$(D_x + a_1 - b_1)\,\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1+1,k_2,l_2} = (a_1-b_1)\tau_{k_1+1,l_1+1,k_2,l_2}\tau_{k_1,l_1,k_2,l_2}$$
$$(D_x + a_2 - b_2)\,\tau_{k_1,l_1,k_2+1,l_2}\cdot\tau_{k_1,l_1,k_2,l_2+1} = (a_2-b_2)\tau_{k_1,l_1,k_2+1,l_2+1}\tau_{k_1,l_1,k_2,l_2}$$
$$(D_x + a_1 - a_2)\,\tau_{k_1+1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2+1,l_2} = (a_1-a_2)\tau_{k_1+1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2,l_2}$$

$b_1 = -a_1, b_2 = -a_2,\ k_1 = l_1 = k_2 = l_2 = 0$

$$\Longrightarrow \qquad \begin{aligned}(D_x + 2a_1)\,\tau_{1000}\cdot\tau_{0100} &= 2a_1\tau_{1100}\tau_{0000}\\ (D_x + 2a_2)\,\tau_{0010}\cdot\tau_{0001} &= 2a_2\tau_{0011}\tau_{0000}\\ (D_x + a_1 - a_2)\,\tau_{1000}\cdot\tau_{0010} &= (a_1-a_2)\tau_{1010}\tau_{0000}\end{aligned}$$

$a_1 = \mathrm{i}\alpha_1,\ a_2 = \mathrm{i}\alpha_2$

$$\Longrightarrow \qquad \begin{aligned}(D_x+2\mathrm{i}\alpha_1)g_1\cdot g_1^* &= 2\mathrm{i}\alpha_1 q_{11} f\\ (D_x+2\mathrm{i}\alpha_2)g_2\cdot g_2^* &= 2\mathrm{i}\alpha_2 q_{22} f\\ (D_x+\mathrm{i}\alpha_1-\mathrm{i}\alpha_2)g_1\cdot g_2 &= (\mathrm{i}\alpha_1-\mathrm{i}\alpha_2)s_{12} f\end{aligned}$$

# KP-Toda Reduction (II)

$$\left(D_{r_1}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1+1,l_1,k_2,l_2}\tau_{k_1-1,l_1,k_2,l_2}$$
$$\left(D_{s_1}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1,l_1+1,k_2,l_2}\tau_{k_1,l_1-1,k_2,l_2}$$
$$\left(D_{r_2}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2-1,l_2}$$
$$\left(D_{s_2}D_x - 2\right)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2\tau_{k_1,l_1,k_2,l_2+1}\tau_{k_1,l_1,k_2,l_2-1}$$

Under dimension reduction condition

$$\left(\partial_{r_1}+\partial_{s_1}+\partial_{r_2}+\partial_{s_2}\right)\tau_{k_1,l_1,k_2,l_2} = \frac{1}{c}\partial_x\tau_{k_1,l_1,k_2,l_2}.$$

$$(D_x^2-8c)\tau_{k_1,l_1,k_2,l_2}\cdot\tau_{k_1,l_1,k_2,l_2} = -2c\Big(\tau_{k_1+1,l_1,k_2,l_2}\tau_{k_1-1,l_1,k_2,l_2}$$
$$+\tau_{k_1,l_1+1,k_2,l_2}\tau_{k_1,l_1-1,k_2,l_2}+\tau_{k_1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2-1,l_2}+\tau_{k_1,l_1,k_2,l_2+1}\tau_{k_1,l_1,k_2,l_2-1}\Big)$$

$k_1 = l_1 = k_2 = l_2 = 0$

$$(D_x^2-8c)\tau_{0000}\cdot\tau_{0000} = -2c\,(\tau_{1000}\tau_{-1000}$$
$$+\tau_{0100}\tau_{0,-1,00}+\tau_{0010}\tau_{00,-1,0}+\tau_{0001}\tau_{000,-1})$$

$$(D_x^2-8c)f\cdot f + 4cg_1g_1^* + 4cg_2g_2^* = 0$$

# KP-Toda Reduction (III)

How about

$$\begin{aligned}&(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2 + 8c)D_x - 12\mathrm{i}c\alpha_1)g_1 \cdot f \\ &+6c[\mathrm{i}\alpha_1 q_{11} g_1 + \mathrm{i}\alpha_2 q_{22} g_1 + \mathrm{i}(\alpha_1 - \alpha_2) s_{12} g_2^*] = 0,\end{aligned} \tag{23}$$

# KP-Toda Reduction (III)

How about

$$
\begin{aligned}
&(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3(\alpha_1^2 + 8c)D_x - 12\mathrm{i}c\alpha_1)g_1 \cdot f \\
&+6c[\mathrm{i}\alpha_1 q_{11} g_1 + \mathrm{i}\alpha_2 q_{22} g_1 + \mathrm{i}(\alpha_1 - \alpha_2)s_{12} g_2^*] = 0,
\end{aligned} \tag{23}
$$

$$\left(D_x^2 - D_y + 2a_1 D_x\right) \tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1,k_2,l_2} = 0 \tag{24}$$

$$\left(D_x^3 + 3D_x D_y - 4D_t + 3a_1 \left(D_x^2 + D_y\right) + 6a_1^2 D_x\right) \tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1,k_2,l_2} = 0 \tag{25}$$

$$\left(D_{r_1}\left(D_x^2 - D_y + 2a_1 D_x\right) - 4D_x\right) \tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1,k_2,l_2} = 0 \tag{26}$$

$$
\begin{aligned}
&\left(D_{s_1}\left(D_x^2 - D_y + 2a_1 D_x\right) - 4\left(D_x + a_1 - b_1\right)\right) \tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1,k_2,l_2} \\
&+4(a_1 - b_1)\tau_{k_1+1,l_1+1,k_2,l_2} \cdot \tau_{k_1,l_1-1,k_2,l_2} = 0
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\left(D_{r_2}\left(D_x^2 - D_y + 2a_1 D_x\right) - 4\left(D_x + a_1 - a_2\right)\right) \tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1,k_2,l_2} \\
&+4(a_1 - a_2)\tau_{k_1+1,l_1,k_2+1,l_2} \cdot \tau_{k_1,l_1,k_2-1,l_2} = 0
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&\left(D_{s_2}\left(D_x^2 - D_y + 2a_1 D_x\right) - 4\left(D_x + a_1 - b_2\right)\right) \tau_{k_1+1,l_1,k_2,l_2} \cdot \tau_{k_1,l_1,k_2,l_2} \\
&+4(a_1 - b_2)\tau_{k_1+1,l_1,k_2,l_2+1} \cdot \tau_{k_1,l_1,k_2,l_2-1} = 0
\end{aligned} \tag{29}
$$

Under dimension reduction condition $\left(\partial_{r_1} + \partial_{s_1} + \partial_{r_2} + \partial_{s_2}\right) \tau_{k_1,l_1,k_2,l_2} = \frac{1}{c}\partial_x \tau_{k_1,l_1,k_2,l_2}$. and

$$3a_1 \times (24) + (25) + 3c \times [(26) + (27) + (28) + (29)]$$

we have

$$
\begin{aligned}
&\left(D_x^3 - D_t + 3\mathrm{i}\alpha_1 D_x^2 - 3\left(\alpha_1^2 + 4c\right) D_x - 12\mathrm{i}\alpha_1 c\right) g_1 \cdot f \\
&+c[6a_1 q_{11} g_1 + 3(a_1 - a_2)s_{12} g_2^* + 3(a_1 + a_2)\tau_{1001} g_2] = 0
\end{aligned} \tag{30}
$$

# KP-Toda Reduction (IV)

(30)-(23) gives

$$(a_1 + a_2)\tau_{1001}g_2 = 2a_2q_{22}g_1 + (a_1 - a_2)s_{12}g_2^* \tag{31}$$

(30)-(23) gives

$$(a_1 + a_2)\tau_{1001} g_2 = 2a_2 q_{22} g_1 + (a_1 - a_2) s_{12} g_2^* \tag{31}$$

Discrete KP (Hirota-Miwa) equation for triples $k_1(a_1), k_2(a_2), l_2(b_2)$

$$(b_2 - a_1)\tau_{k_1+1,l_1,k_2,l_2+1}\tau_{k_1,l_1,k_2+1,l_2} + (a_2 - b_2)\tau_{k_1,l_1,k_2+1,l_2+1}\tau_{k_1+1,l_1,k_2,l_2}$$
$$+(a_1 - a_2)\tau_{k_1+1,l_1,k_2+1,l_2}\tau_{k_1,l_1,k_2,l_2+1} = 0$$

If $b_2 = -a_2$, $k_1 = l_1 = k_2 = l_2 = 0$, we have

$$(-a_2 - a_1)\tau_{1001}\tau_{0010} + 2a_2\tau_{0011}\tau_{1000} + (a_1 - a_2)\tau_{1010}\tau_{0001} = 0$$

which is exactly (31).

# $N$-breather solution

To derive regular breather solution, we take $c_{ij} = 0$, $K = 2$

$$\tau_{k_1,l_1,k_2,l_2} = \det\left(m_{ij}^{k_1,l_1,k_2,l_2}\right)_{1\le i,j\le N}$$

where the matrix element is defined as

$$m_{ij}^{k_1,l_1,k_2,l_2} = \sum_{m,n=1}^{2} \frac{1}{p_{im}+\bar{p}_{jn}}\left(-\frac{p_{im}-a_1}{\bar{p}_{jn}+a_1}\right)^{k_1}\left(-\frac{p_{im}-b_1}{\bar{p}_{jn}+b_1}\right)^{l_1}$$
$$\left(-\frac{p_{im}-a_2}{\bar{p}_{jn}+a_2}\right)^{k_2}\left(-\frac{p_{im}-b_2}{\bar{p}_{jn}+b_2}\right)^{l_2} e^{\xi_{im}+\bar{\xi}_{jn}}$$

By imposing the condition $H(p_{i1}, p_{i2}) = 0$, where

$$H(p_{i1}, p_{i2}) = \frac{p_{i1}p_{i2}+a_1^2}{\left(p_{i1}^2-a_1^2\right)\left(p_{i2}^2-a_1^2\right)} + \frac{p_{i1}p_{i2}+a_2^2}{\left(p_{i1}^2-a_2^2\right)\left(p_{i2}^2-a_2^2\right)} + \frac{1}{2c},$$

we then have

$$\left(\partial_{r_1}+\partial_{s_1}+\partial_{r_2}+\partial_{s_2}\right)\sigma_{k_1,l_1,k_2,l_2} = \frac{1}{c}\partial_x\sigma_{k_1,l_1,k_2,l_2}.$$

# General breather solution including resonant breather solution

## Theorem 2.

*The coupled Sasa-Satsuma (SS) equation admits the general breather solutions*

$$u_1 = \frac{g_1}{f} e^{\mathrm{i}\left(\alpha_1(x-12ct)-\alpha_1^3 t\right)}, \quad u_2 = \frac{g_2}{f} e^{\mathrm{i}\left(\alpha_2(x-12ct)-\alpha_2^3 t\right)} \tag{32}$$

*where* $\alpha_1, \alpha_2$ *are real,*

$$f(x,t) = \tau_{00}(x-12ct,t), \quad g_1(x,t) = \tau_{10}(x-12ct,t), \quad g_2(x,t) = \tau_{01}(x-12ct,t)$$

*and* $\tau_{k_1,k_2}$ *is defined as*

$$\tau_{k_1,k_2} = \left| \sum_{m,n=1}^{2\le K\le 5} \frac{1}{p_{im}+p_{jn}} \left(-\frac{p_{im}-a_1}{p_{jn}+a_1}\right)^{k_1} \left(-\frac{p_{im}-a_2}{p_{jn}+a_2}\right)^{k_2} e^{\xi_{im}+\xi_{jn}} \right|_{2M\times 2M}. \tag{33}$$

*Here,* $a_1 = \mathbf{i}\alpha_1, a_2 = \mathbf{i}\alpha_2$ *are purely imaginary,* $\xi_{im} = p_{im}x + p_{im}^3 t + \xi_{im,0}$, $N$ *is a positive integer. In addition, the parameters* $\xi_{jm,0} \in \mathbb{R}, p_{im}, \ (i,j = 1,\cdots,N; m,n = 1,2)$ *satisfy the constraints*

$$H(p_{im}, p_{iK}) = 0, \quad m = 1,\cdots,K-1, \quad i = 1,2,\ldots,M, \tag{34}$$

$$H(p,q) = \frac{pq+a_1^2}{\left(p^2-a_1^2\right)\left(q^2-a_1^2\right)} + \frac{pq+a_2^2}{\left(p^2-a_2^2\right)\left(q^2-a_2^2\right)} + \frac{1}{2c}. \tag{35}$$

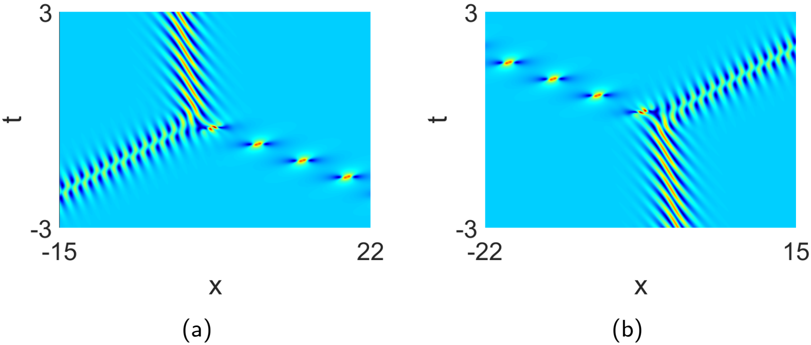
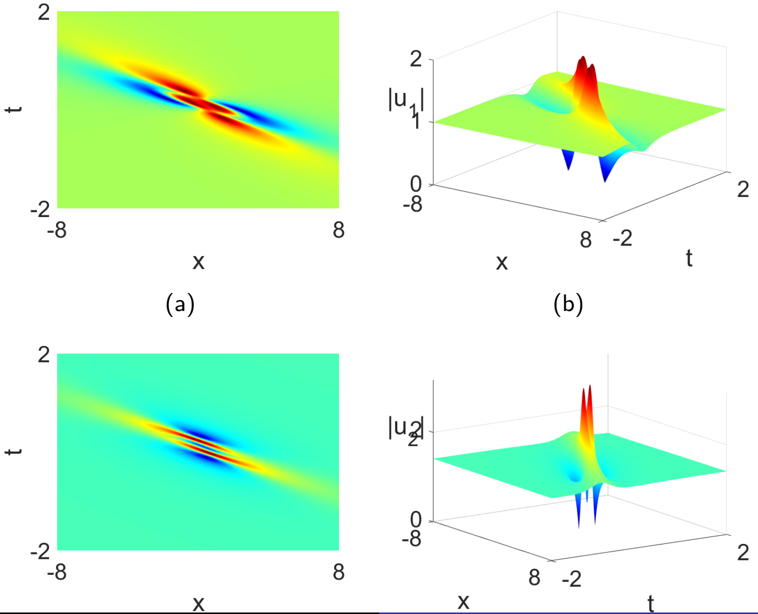
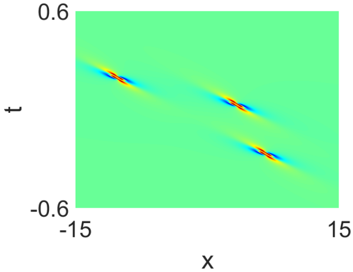


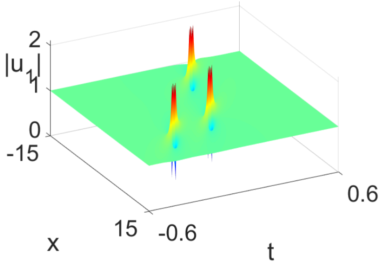

Figure: Resonant breather solution with parameters $c = -1, a = 2, p1 = 1.2 + 1.6i$.

# Rogue wave solution

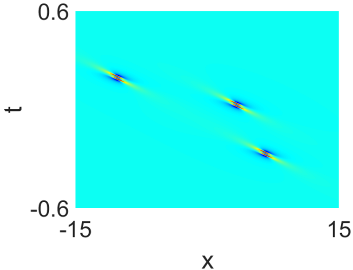


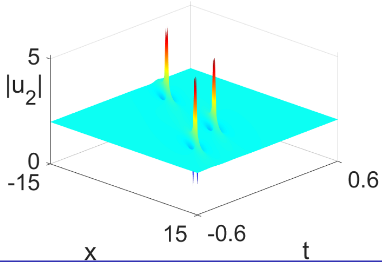
# Second-order RW solution




(g) (h)



Thank you!
Questions?