A hydrodynamic formulation for a nonlinear Dirac equation

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Energy, Entropy, and Dissipative Dynamics

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Table of contents

Motivation

Hydrodynamic formulation for the Schrödinger equation

$$i\partial_t \psi(t,x) = \left(\frac{-1}{2m}\Delta + V(t,x)\right)\psi(t,x)$$

can be rewritten as a system of conservation laws in terms of hydrodynamics variables: if

$$\psi(t,x) =: \sqrt{\frac{1}{m}\varrho(t,x)}\,\mathsf{exp}(iS(t,x)),$$

then ϱ and $u := m^{-1}\nabla S(t,x)$ satisfy the compressible Euler type equations

$$\partial_t \varrho + \mathsf{div}(\varrho u) = 0$$

$$\partial_t u + u \cdot \nabla u + \frac{1}{m}\nabla(V + Q) = 0$$

The quantum effects are all captured in Bohm's potential

$$Q := -\frac{1}{2m}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}$$

Motivation

Hydrodynamic formulation for the Dirac equation

- There has been substantial work trying to establish a hydrodynamics formulation of the Dirac equation, most notably by Takabayasi.
- We propose a different hydrodynamics formulation
  - using of the **space algebra** instead of the more commonly used $\mathbf{C}^4$-valued spinor functions
  - and considering a **nonlinear** variant of the Dirac equation.
- The model was proposed by Daviau and
  1. adds the minimal amount of nonlinearity needed to achieve an additional $U(1)$ symmetry while keeping the first-order homogeneity,
  2. admits a natural splitting of the spinor into left and right-handed components, which is crucial for our approach, and
  3. can correctly predict the energy levels in a hydrogen atom (C. Daviau et al. 2020).

Space algebra

- The space algebra Cl_3 is the Clifford algebra of the three-dimensional Euclidean space $\mathbf{R}^3$. It adds to the usual vector calculus an associative product with the following fundamental property

$$u^2 = uu = u \cdot u \text{ for any } u \in \mathbf{R}^3,$$

  i.e. the square of a vector gives the Euclidean inner product of the vector with itself.
- Expanding vectors $u \in \mathbf{R}^3$ in terms of an orthonormal basis $e_k, k = 1 \dots 3$, this requirements translates into the structure equation

$$e_k e_l + e_l e_k = 2\delta_{kl},$$

  for $k = 1 \dots 3$. The most straightforward and non-intrusive way to represent such a product is to use matrix multiplication and the Pauli matrices

$$e_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Space algebra

- Elements in $\mathbf{R}^3$ are associated with it a corresponding element in Cl_3,

$$x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbf{R}^3 \Longleftrightarrow \vec{x} = x^k e_k = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix} \in \mathrm{Cl}_3.$$

- With the additional basis vector **1** we can embed spacetime into Cl_3,

$$\begin{pmatrix} x^0 \\ x \end{pmatrix} \in \mathbf{R} \times \mathbf{R}^3 \Longleftrightarrow x^\mu e_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \in \mathrm{Cl}_3.$$

- Vectors in spacetime are therefore represented in Cl_3 by linear combinations of basis vectors e_μ with real coefficients i.e. **Hermitian matrices** e.g.

  the proper velocity $u = \gamma\mathbf{1} + \vec{u} = \gamma(\mathbf{1} + \vec{v})$, the charge current $j = j^0 + \vec{j}$,
  the electromagnetic potential $A = A_0 + \vec{A}$, the energy-momentum $p = E + \vec{p}, \ldots$

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Space algebra

- A basis of Cl_3 is given by

$$\{e_0 := \mathbf{1}, e_k, e_k e_j, e_1 e_2 e_3\} \quad \text{with } 1 \leqslant k < j \leqslant 3,$$

  therefore Cl_3 is 8-dimensional vector space over **R**. This is the same dimension as for Dirac spinors, which classically are represented as $\mathbf{C}^4$-valued vector fields.
- It is not hard to check that $e_1 e_2 e_3$ squares to -1 and commutes with all other basis vectors. Identifying $e_1 e_2 e_3$ with the imaginary unit i, it is convenient to turn Cl_3 into a vector space over the complex numbers **C** i.e. to consider

$$M = a + \vec{u} + i\vec{v} + ib \quad \text{with } a, b \in \mathbf{R} \text{ and } u, v \in \mathbf{R}^3.$$

  Thus M is a sum of a spacetime vector $a + \vec{u}$ and i times another spacetime vector $b + \vec{v}$.

Space algebra

Inner product in Cl_3

- Since elements in Cl_3 are matrices, we can use the Frobenius inner product

$$\langle M, N\rangle := \frac{1}{2}\mathrm{tr}(MN^\dagger) \quad \text{for } M, N \in \mathrm{Cl}_3,$$

  and denote by $\|M\| := \langle M, M\rangle^{1/2}$ the induced norm.
- The basis vectors e_μ are orthonormal with respect to this inner product.
- For all vectors $x = x^0 + \vec{x}, y = y^0 + \vec{y} \in \mathrm{Cl}_3$ we have that

$$\langle x, y\rangle = x^0 y^0 + x \cdot y \text{ and } \|x\|^2 = (x^0)^2 + \|x\|^2.$$

  In this sense, the embedding of spacetime $\mathbf{R} \times \mathbf{R}^3$ into Cl_3 is an isometry.

Space algebra

Three conjugations

$$M =: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \iff M^\dagger = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$$ Hermitian conjugation

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \iff \bar{M} := \text{adj}(M) = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$ spatial reversal

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \iff \hat{M} := \bar{M}^\dagger = \begin{pmatrix} D^* & -C^* \\ -B^* & A^* \end{pmatrix}$$ grade automorphism

The gradient operator

- The space algebra analogue of the gradient operator is

$$\nabla := e^\mu \partial_\mu = \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix}.$$

- When ∇ is applied to a scalar-valued function f, then we have that

$$\nabla f = (\partial_0 f)\, e^0 + (\partial_1 f)\, e^1 + (\partial_2 f)\, e^2 + (\partial_3 f)\, e^3,$$

  but, in general, when applied to a Cl_3-valued function ϕ, then the e^μ interact with the function ϕ by **matrix multiplication**.
- We can split the operator ∇ into a time (scalar) and a spatial (vector) part, writing

$$\nabla = \partial_0 + \vec{\nabla} \quad \text{with} \quad \vec{\nabla} = e^k \partial_k$$

Improved Dirac equation

We start with the classical Dirac equation

$$\gamma^\mu D_\mu \psi + im\psi = 0$$

where the covariant derivative $D_\mu := \partial_\mu + iqA_\mu$ couples ψ to an external electromagnetic field, with A_μ a given electromagnetic potential.
We work in the **chiral representation of the gamma matrices**, thus

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & e^0 \\ e^0 & 0 \end{pmatrix}, \quad \gamma^k = -\gamma_k = \begin{pmatrix} 0 & e^k \\ -e^k & 0 \end{pmatrix}, k = 1 \dots 3,$$

and let

$$\psi =: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{with} \quad \xi =: \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \eta =: \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Improved Dirac equation

Block structure

- Then the block structure of the Dirac equation implies that

$$e^{\mu}(\partial_{\mu} + iqA_{\mu})\eta + im\xi = 0 \qquad \hat{e}^{\mu}(\partial_{\mu} + iqA_{\mu})\xi + im\eta = 0$$

- We can combine these two equations into a single one by interpreting these two equations as the left/right column vectors of a (2×2)-matrix. More precisely, let

$$L := \begin{pmatrix} 0 & -\eta_2^* \\ 0 & \eta_1^* \end{pmatrix} \quad \text{and} \quad R := \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix}$$

be the left/right parts of the spinor wave so that

$$(\nabla + iqA)\hat{L} + imR = 0 \qquad (\nabla - iqA)\hat{R} - imL = 0$$

- We can therefore combine them by adding them:

$$\nabla\hat{\phi} + iqA\hat{\phi}e_3 + im\phi e_3 = 0$$

for the Cl_3-valued spinor field $\phi := L + R$.

Improved Dirac equation

- If we multiply by the spatial reversal $\bar{\phi}$, it reads

$$\bar{\phi}\nabla\hat{\phi} + iq(\bar{\phi}A\hat{\phi})e_3 + im(\bar{\phi}\phi)e_3 = 0.$$

- But this equation is not invariant under transformations of the form

$$\phi(x) = M^{-1}\psi(y) \quad \text{with} \quad y = MxM^\dagger,$$

for some invertible complex (2×2)-matrix M (not necessarily with det $M = 1$)
- E.g. if $M = \exp(i\beta/2)\mathbf{1}$ with $\beta \neq 0$ then $\det(M) = \exp(i\beta)$ and

$$\bar{\phi}(x)\phi(x) = \bar{\psi}(y)(\bar{M}M)\psi(y) = \det(M)\bar{\psi}(y)\psi(y).$$

This is strange because for this choice of M, spacetime does not change at all:

$$y = MxM^\dagger = x$$

only ψ is multiplied by some global exponential factor.

Improved Dirac equation

- Following Daviau, we consider the nonlinear Dirac equation

$$\bar{\phi}\nabla\hat{\phi} + iq(\bar{\phi}A\hat{\phi})e_3 + imPe_3 = 0 \quad \text{with} \quad P := |\,\mathrm{det}(\phi)|.$$

- The nonlinearity has the same homogeneity as the original term $\bar{\phi}\phi$.
- We now define the Dirac current $J := \phi\phi^\dagger$ and obtain

$$P\mathbf{1} = \bar{\phi}V\hat{\phi} \quad \text{with velocity} \quad V := P^{-1}J.$$

$$P^2 = |\,\mathrm{det}(\phi)|^2 = \frac{1}{2}\mathrm{tr}\left((\bar{\phi}\phi)(\bar{\phi}\phi)^\dagger\right) = \frac{1}{2}\mathrm{tr}\left((\phi\phi^\dagger)\,\overline{(\phi\phi^\dagger)}\right) = \mathrm{det}(J)$$

- The nonlinear Dirac equation takes the form

$$\bar{\phi}\nabla\hat{\phi} + i\bar{\phi}(qA + mV)\hat{\phi}e_3 = 0.$$

Assuming that ϕ is invertible, we obtain the remarkably simple equation

$$\nabla\hat{\phi} + i(qA + mV)\hat{\phi}\sigma_3 = 0$$

Global existence

$$\begin{cases} \nabla\hat{\phi} + i(qA + mV)\hat{\phi}\sigma_3 = 0 & \text{in } [0,T]\times \mathbf{R}^3 \\ \phi|_{t=0} = \phi_0 & \text{in } \mathbf{R}^3 \end{cases}$$

- Regularisation becomes necessary because the nonlinearity, which is homogeneous of degree one, is non-smooth in nodal points where P vanishes.
- Our approach replaces the $imV\hat{\phi}\sigma_3 = imP^{-1}J\hat{\phi}\sigma_3$ with

$$im(P + \lambda\|\phi\|^2)^{-1}J\hat{\phi}\sigma_3 =: g_\lambda(\phi)$$

where $\|\psi\|^2 := \psi^\dagger\psi$ and $\lambda > 0$. Then

– maps $L^2(\mathbf{R}^3)^4$ into itself and

$$\|g_\lambda(\phi)\|_{L^2(\mathbf{R}^3)^4} \lesssim m\|\phi\|_{L^2(\mathbf{R}^3)^4} \quad \text{for all} \quad \phi \in L^2(\mathbf{R}^3)^4$$

– is Lipschitz continuous on $L^2(\mathbf{R}^3)^4$,

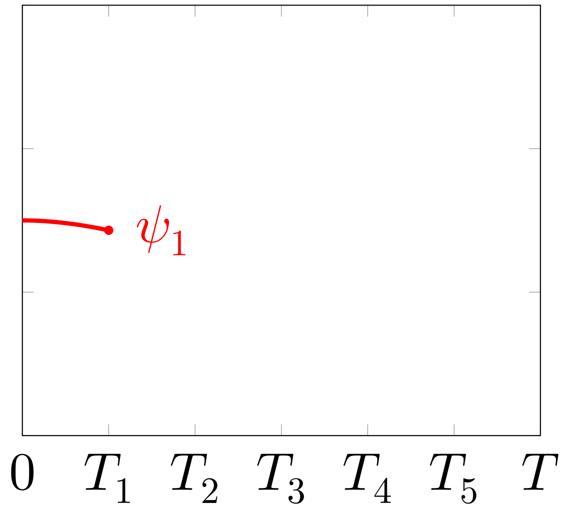
$$\|g_\lambda(\phi) - g_\lambda(\phi')\|_{L^2(\mathbf{R}^3)^4} \lesssim m(\lambda^{-1}+1)\|\phi - \phi'\|_{L^2(\mathbf{R}^3)^4} \quad \text{for all} \quad \phi, \phi' \in L^2(\mathbf{R}^3)^4$$

– maps $H^1(\mathbf{R}^3)^4$ into itself and

$$\|g_\lambda(\phi)\|_{H^1(\mathbf{R}^3)^4} \lesssim m(\lambda^{-1}+1)\|\phi\|_{H^1(\mathbf{R}^3)^4} \quad \text{for all} \quad \phi \in H^1(\mathbf{R}^3)^4$$

Global existence

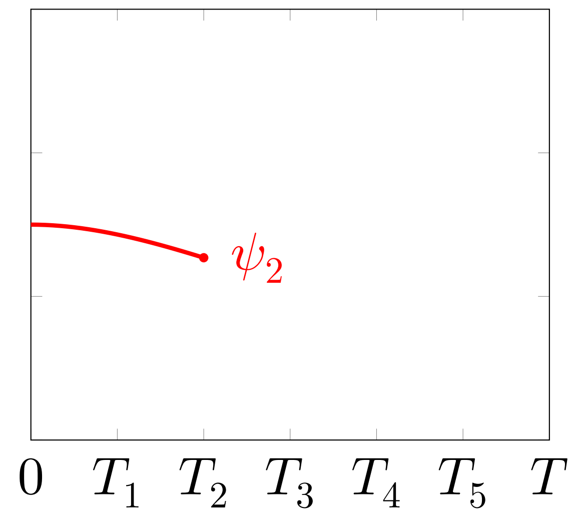
Time-stepping



$$\begin{cases} \nabla\hat{\psi} - g_\lambda(\phi_0) = 0 \quad \text{in } [0,T_1]\times \mathbf{R}^3 \\ \psi|_{t=0} = \phi_0 \quad \text{in } \mathbf{R}^3 \\ \psi \in C([0,T_1], H^1(\mathbf{R}^3)^4) \cap C^1([0,T_1], L^2(\mathbf{R}^3)^4) \end{cases}$$

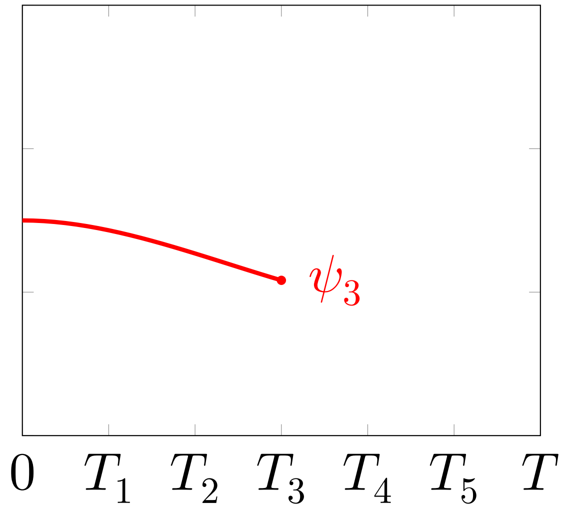
Global existence

Time-stepping



$$\begin{cases} \nabla\hat{\psi} - g_\lambda(\psi_1(T_1)) = 0 \quad \text{in } [T_1, T_2] \times \mathbf{R}^3 \\ \psi|_{t=T_1} = \psi_1(T_1) \quad \text{in } \mathbf{R}^3 \\ \psi \in C([T_1, T_2], H^1(\mathbf{R}^3)^4) \cap C^1([T_1, T_2], L^2(\mathbf{R}^3)^4) \end{cases}$$

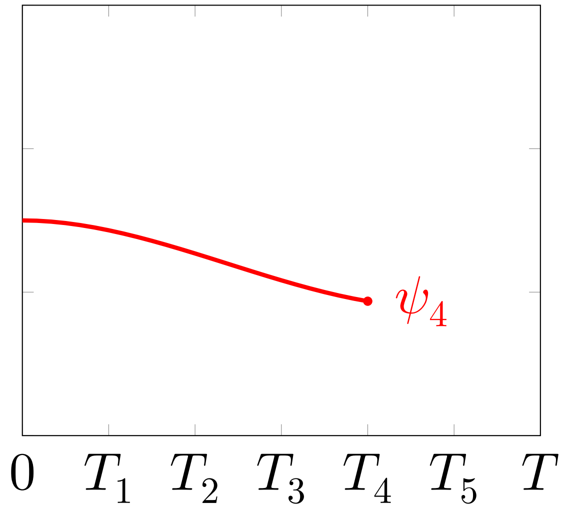
Time-stepping



$$\begin{cases} \nabla\hat{\psi} - g_\lambda(\psi_2(T_2)) = 0 & \text{in } [T_2, T_3] \times \mathbf{R}^3 \\ \psi|_{t=T_2} = \psi_2(T_2) & \text{in } \mathbf{R}^3 \\ \psi \in C([T_2, T_3], H^1(\mathbf{R}^3)^4) \cap C^1([T_2, T_3], L^2(\mathbf{R}^3)^4) \end{cases}$$

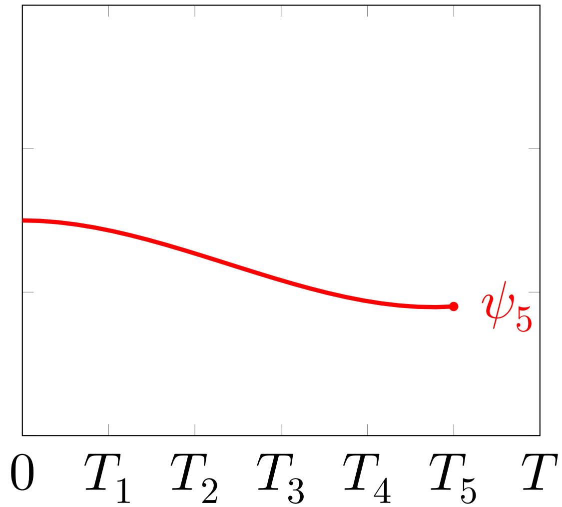
Global existence

Time-stepping



$$\begin{cases} \nabla\hat{\psi} - g_\lambda(\psi_3(T_3)) = 0 \quad \text{in } [T_3, T_4] \times \mathbf{R}^3 \\ \psi|_{t=T_3} = \psi_3(T_3) \quad \text{in } \mathbf{R}^3 \\ \psi \in C([T_3, T_4], H^1(\mathbf{R}^3)^4) \cap C^1([T_3, T_4], L^2(\mathbf{R}^3)^4) \end{cases}$$

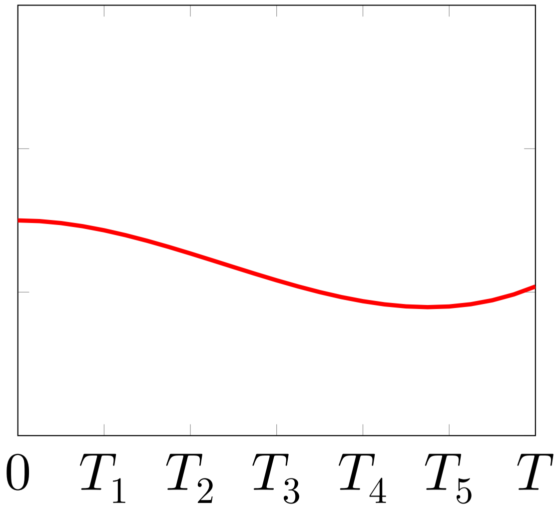
Time-stepping



$$\begin{cases} \nabla\hat{\psi} - g_\lambda(\psi_4(T_4)) = 0 \quad \text{in } [T_4, T_5] \times \mathbf{R}^3 \\ \psi|_{t=T_2} = \psi_4(T_4) \quad \text{in } \mathbf{R}^3 \\ \psi \in C([T_4, T_5], H^1(\mathbf{R}^3)^4) \cap C^1([T_4, T_5], L^2(\mathbf{R}^3)^4) \end{cases}$$

Global existence

Time-stepping



$$\begin{cases} \nabla\hat{\psi} - g_\lambda(\psi_5(T_5)) = 0 \quad \text{in } [T_5, T] \times \mathbf{R}^3 \\ \psi|_{t=T_5} = \psi_5(T_5) \quad \text{in } \mathbf{R}^3 \\ \psi \in C([T_5, T], H^1(\mathbf{R}^3)^4) \cap C^1([T_5, T], L^2(\mathbf{R}^3)^4) \end{cases}$$

Compactness and consistency of the approximate solutions

- Moreover, for every $1 \leq k \leq n$, we have the estimates
$$\|\psi_k(t)\|_{H^1(\mathbf{R}^3)^4} \leq \|\psi_{k-1}(T_{k-1})\|_{H^1(\mathbf{R}^3)^4}(1 + C_{m,\lambda}(t - T_{k-1}))$$
with $C_{m,\lambda} \propto m(\lambda^{-1} + 1)$, for all $t \in [T_{k-1}, T_k]$.
- So if we define $\{\phi_n(t)\}_{n=1}^{\infty}$ piece-wise as
$$\phi_n(t) := \psi_k(t) \quad \text{if } t \in [T_{k-1}, T_k],$$
for every $1 \leq k \leq n$ then,
$$\{\phi_n(t)\}_{n=1}^{\infty} \text{ is bounded uniformly in } L^\infty([0,T], H^1(\mathbf{R}^3)^4)$$
$$\text{and } \{\partial_t \phi_n(t)\}_{n=1}^{\infty} \text{is bounded uniformly in } L^\infty([0,T], L^2(\mathbf{R}^3)^4).$$
- A **compactness result** (Aubin-Lions lemma) shows that the limit $\phi(t)$ is a weak solution in the sense that $\phi(t) \in L^\infty([0,T], L^2(\mathbf{R}^3)^4)$ and
$$\int_0^T \int_{\mathbf{R}^3} \langle \nabla \xi(t), \phi(t) \rangle d\vec{x} dt - i \int_{\mathbf{R}^3} \langle \xi(0), \phi_0 \rangle d\vec{x} = \int_0^T \int_{\mathbf{R}^3} \langle \xi(t), g_\lambda(\phi(t) \rangle d\vec{x} dt$$
for every test function $\xi \in C_c^\infty([0,T) \times \mathbf{R}^3)$.

Hydrodynamics formulation

Lemma (1)

Suppose the **C**^{2}*-valued spinor field* η *satisfies*

$$\nabla\eta + i(qA + mV)\eta = 0 \quad \textit{in } [0,\infty) \times \mathbf{R}^3,$$

with electromagnetic potential A *and a given Hermitian vector field* V, *to which we will refer as pilot wave. We define the energy-momentum tensor by*

$$T_{\mu\nu} := \mathrm{Re}(i(\eta^\dagger e^\mu (D_\nu \eta))) \quad \textit{for } \mu, \nu = 0 \ldots 3.$$

Then

$$\partial_\mu T_{\mu\nu} + qF_{\mu\nu} j_\mu = m\left(\partial_\nu V_\mu\right) j_\mu \quad \textit{for } \nu = 0 \ldots 3,$$

with the charge current j *and electromagnetic tensor field* F *defined by*

$$j := j_\mu e^\mu := 2\eta\eta^\dagger \quad \textit{and} \quad F := F_{\mu\nu} e^\mu e^\nu, \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Hydrodynamics formulation

Lemma (2)

Let η and $j_\mu, T_{\mu\nu}$ be as in the lemma on the previous slide. Then $\partial_\mu j_\mu = 0$ and

$$\partial_0 j_k + \partial_k j_0 + 2\epsilon_{kln} T_{ln} = 2m\epsilon_{kln} V_n j_l$$

for $k = 1 \ldots 3$.

Lemma (3)

Let η and $j_\mu, T_{\mu\nu}$ be as in the lemma on the previous slide and define

$$\varrho := j_0, \quad v_k := \frac{j_k}{j_0} \quad \text{and} \quad p_k := T_{0k} \quad \text{for} \quad k = 1 \ldots 3.$$

Then $\|v\| = 1$ and the energy-momentum tensor can be rewritten as

$$T_{ln} = v_l p_n - \frac{1}{2}\epsilon_{lko} v_k \left(\partial_n j_o\right) \qquad \text{for } l, n = 1 \ldots 3.$$

Hydrodynamics formulation

Theorem

Consider a spinor ϕ satisfying the nonlinear Dirac equation. Let

$$\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \hat{\xi} := \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} \quad \textit{with} \quad \hat{\phi} =: \begin{pmatrix} \eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^* \end{pmatrix}.$$

Define $(\varrho_\cdot, v_\cdot, p_\cdot, j_\cdot)$ for $\cdot = L, R$ as before for $\eta, \hat{\xi}$, respectively. Then

$$\begin{cases} \partial_t \varrho + \text{div}(\varrho v) = 0 \\ \partial_t(\varrho v) + \text{div}(\varrho v \otimes v) + 2\varrho(v \times (\nabla \times v)) - (v \times (v \times \nabla \varrho)) = 2v \times (-p \pm m\varrho V) \\ \partial_t p_n + \text{div}\,(p_n v) - \frac{1}{2}\text{div}\,(\varrho v \times (\partial_n v)) = \pm(-qF_{\mu n} + m(\partial_n V_\mu))j_\mu \end{cases}$$

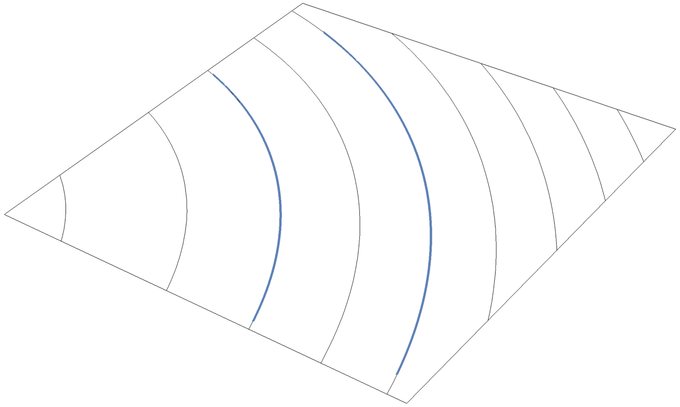
for $n = 1 \ldots 3$, with sign

$$\begin{aligned} + \quad & \text{for}(\varrho, v, p, j) = (\varrho_L, v_L, p_L, j_L)\,, \\ - \quad & \textit{for}\ (\varrho, v, p, j) = (\varrho_R, v_R, p_R, j_R)\,. \end{aligned}$$

Hydrodynamics formulation

- The picture suggested is that for a solution ϕ to the nonlinear Dirac equation, $J = \phi\phi^\dagger$ defines the **flowlines that guide the motion of the left and right spinors** (i.e. their flowlines curl helically around the flowlines of the Dirac current). We call the Dirac current the pilot wave, using De Broglie's terminology.
- The left and right spinor flowlines move with light speed as $\|v\| = 1$, whereas the velocity of the Dirac current is subluminal.

Flowlines of the Dirac current J



Hydrodynamics formulation

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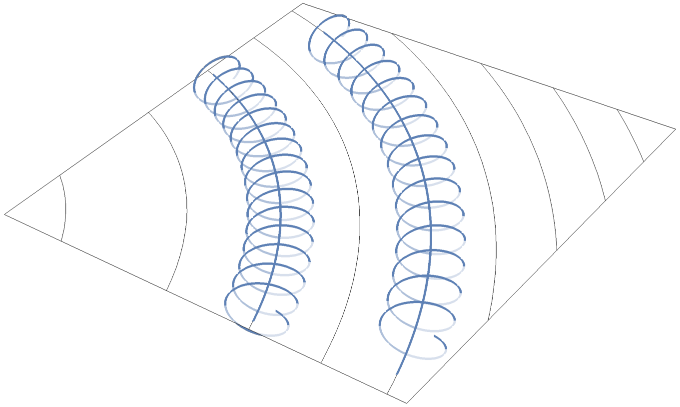
- The left and right spinor flowlines move with light speed as $\|v\| = 1$, whereas the velocity of the Dirac current is subluminal.

Flowlines of the charge current j



Thank you for your attention

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Multiplicative inverses

- One big advantage of using a Clifford algebra comes from the fact that elements in Cl_3 may have multiplicative inverses.
- Elements in Cl_3 with non-vanishing determinant are thus invertible with

$$M^{-1} = (M\bar{M})^{-1}\bar{M}$$

which functions both as a left and right inverse because $M\bar{M} = \bar{M}M$. Notice that det(M) is typically different from the norm $\|M\|^2$.

Fierz identity

We can use the scalar product $[M, N] := \frac{1}{2}\text{tr}(M\overline{N})$ to compute the coefficients of elements in Cl_3 with respect to the basis vectors.

Lemma

For any $M \in \text{Cl}_3$ *we have*

$$M = [M, e^\mu]\, e_\mu = \left[M, e_\mu\right] e^\mu.$$

In particular, for any $\xi, \eta \in \mathbf{C}^2$ *we have the basic Fierz identity*

$$\eta\xi^\dagger = \left[\eta\xi^\dagger, e_\mu\right] e^\mu = \frac{1}{2}\text{tr}((\eta\xi^\dagger)\overline{e_\mu})e^\mu = \frac{1}{2}(\xi^\dagger e^\mu \eta)e^\mu$$

Proof.

We apply $[\cdot, e^\nu]$ to $M = w^\mu e_\mu$ with $w^\mu \in \mathbf{C}$ and then use that $[e_\mu, e^\nu] = \delta_{\mu\nu}$. □

- Hermitian conjugation can be used to characterise spacetime vectors.
- Spatial reversal can be used to isolate scalar and vector parts of M, defined as

$$\langle M\rangle_{\text{scalar}} := \frac{1}{2}(M+\bar{M}) \quad \text{and} \quad \langle M\rangle_{\text{vector}} := \frac{1}{2}(M-\bar{M}).$$

- Grade automorphism can be used to isolate even and odd parts of M, defined as

$$\langle M\rangle_{\text{even}} := \frac{1}{2}(M+\hat{M}) \quad \text{and} \quad \langle M\rangle_{\text{odd}} := \frac{1}{2}(M-\hat{M}).$$

The determinant of a spacetime vector $x = x^\mu e_\mu$ with $x^\mu \in \mathbf{R}$ is

$$\det(x) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^0)^2 - \|x\|^2$$

which is just the Minkowski metric.

Compactness

- For $0 \leq s < 1$, the family of functions $\mathcal{F} := \{\phi_n(t) : n \geq 1\}$ is pre-compact in $C([0,T], H^s_{\text{loc}}(\mathbf{R}^3)^4)$. Thus, the sequence of approximate solutions $\{\phi_n(t)\}_{n=1}^{\infty}$ admits a subsequence, not relabelled here, such that

$$\phi_n(t) \to \phi(t) \quad \text{strongly in} \quad C([0,T], H^s_{\text{loc}}(\mathbf{R}^3)^4).$$

- Idea of the proof: given a compact set $\mathbf{K} \subseteq \mathbf{R}^3$, $\mathcal{F}$ consists of elements in

$$W := \{\psi(t) \in L^\infty([0,T], H^1(\mathbf{R}^3)^4) : \partial_t \psi(t) \in L^\infty([0,T], L^2(\mathbf{R}^3)^4))\}$$

  Hence, restrictions to $\mathbf{K}$ of elements of $\mathcal{F}_{\mathbf{K}}$ belong to

$$W(\mathbf{K}) := \{\psi(t) \in L^\infty([0,T], H^1(\mathbf{K})^4) : \partial_t \psi(t) \in L^\infty([0,T], L^2(\mathbf{K})^4))\}.$$

  By the Rellich-Kondrachov's theorem, $H^1(\mathbf{K})^4 \hookrightarrow H^s(\mathbf{K})^4$ for any $0 \leq s < 1$. Thus we can apply the Aubin-Lions lemma to see that $W(\mathbf{K})$ is pre-compact in $C([0,T], H^s(\mathbf{K})^4)$.

Consistency of the approximate solutions

$$\int_0^T \int_{\mathbf{R}^3} \langle \nabla \xi(t), \phi(t) \rangle dx dt - i \int_{\mathbf{R}^3} \langle \xi(0), \phi_0 \rangle dx = \int_0^T \int_{\mathbf{R}^3} \langle \xi(t), g_\lambda(\phi(t)) \rangle dx dt$$

Note the second integral term on the left-hand side works as a weak formulation of the fact $\psi(0) = \psi_0$.

- Starting with the LHS, we consider:

$$\int_0^T \int_{\mathbf{R}^3} \langle \nabla \xi(t), \phi_n(t) \rangle dx dt - i \int_{\mathbf{R}^3} \langle \xi(0), \phi_0 \rangle dx$$

- By integrating by parts,

$$\int_0^T \int_{\mathbf{R}^3} \langle \nabla \xi(t), \phi_n(t) \rangle dx dt - i \int_{\mathbf{R}^3} \langle \xi(0), \phi_0 \rangle dx = \int_0^T \int_{\mathbf{R}^3} \langle \xi, \hat{\nabla} \phi_n \rangle d\vec{x} dt$$

$$\int_0^T \int_{\mathbf{R}^3} \langle \xi, \hat{\nabla}\phi_n \rangle d\vec{x}dt = \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} \int_{\mathbf{R}^3} \langle \xi, \hat{\nabla}\psi_k \rangle d\vec{x}dt$$

$$= \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} \int_{\mathbf{R}^3} \langle \xi, g_\lambda(\psi_{k-1}(T_{k-1})) \rangle d\vec{x}dt$$

(by the definition of the ϕ_n) $$= \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} \int_{\mathbf{R}^3} \langle \xi, g_\lambda(\phi_{k-1}(T_{k-1})) \rangle d\vec{x}dt$$

(by the mean value theorem) $$= \frac{T}{n} \sum_{k=1}^{n} \int_{\mathbf{R}^3} \langle \xi(T_{k-1}), g_\lambda(\phi_n(T_{k-1})) \rangle d\vec{x} + o(T/n)$$

using that for each component of ξ

$$\xi_i(t) = \xi_i(T_{k-1}) + \int_{T_{k-1}}^{t} \xi_i'(s)ds \qquad i = 1 \ldots 4.$$

- It remains to pass to the limit:

$$\lim_n \frac{T}{n} \sum_{k=1}^{n} \int_{\mathbf{R}^3} \langle \xi(T_{k-1}), g_\lambda(\phi_n(T_{k-1}))\rangle d\vec{x} = ?$$

which we do by adding and subtracting

$$\frac{T}{n} \sum_{k=1}^{n} \int_{\mathbf{R}^3} \langle \xi(T_{k-1}), g_\lambda(\phi(T_{k-1}))\rangle d\vec{x}$$

- On one hand

$$\lim_n \frac{T}{n} \sum_{k=1}^{n} \int_{\mathbf{R}^3} \langle \xi(T_{k-1}), g_\lambda(\phi_n(T_{k-1})) - g_\lambda(\phi(T_{k-1}))\rangle d\vec{x} = 0.$$

by the strong convergence in $C([0,T], L^2_{\text{loc}}(\mathbf{R}^3)^4)$, and on the other,

$$\lim_n \frac{T}{n} \sum_{k=1}^{n} \int_{\mathbf{R}^3} \langle \xi(T_{k-1}), g_\lambda(\psi(T_{k-1}))\rangle d\vec{x} = \int_0^T \int_{\mathbf{R}^3} \langle \xi(t), g_\lambda(\psi(t))\rangle d\vec{x}dt$$

by Lebesgue's theory of integration.

Time-stepping

- If g is of class $C([T_0,T],L^2(\mathbf{R}^3)^4)\cap L^1([T_0,T],H^1(\mathbf{R}^3)^4)$, then for any initial condition $\psi_0\in H^1(\mathbf{R}^3)^4$ the initial value problem

$$\begin{cases}\nabla\hat{\psi}-g(t)=0 & \text{in } [T_0,T]\times\mathbf{R}^3\\ \psi|_{t=T_0}=\psi_0 & \text{in } \mathbf{R}^3\\ \psi\in C([T_0,T],H^1(\mathbf{R}^3)^4)\cap C^1([T_0,T],L^2(\mathbf{R}^3)^4)\end{cases}$$

  has a unique solution.
- That is, for any $n\geq 0$ the initial value problems

$$\begin{cases}\nabla\hat{\psi}-g_\lambda(\psi_{k-1}(T_{k-1}))=0 & \text{in } [T_{k-1},T_k]\times\mathbf{R}^3\\ \psi|_{t=T_{k-1}}=\phi_{k-1}(T_{k-1}) & \text{in } \mathbf{R}^3\\ \psi\in C([T_{k-1},T_k],H^1(\mathbf{R}^3)^4)\cap C^1([T_{k-1},T_k],L^2(\mathbf{R}^3)^4)\end{cases}$$

  have a unique solution $\psi_k(t)$ and $\psi_k(T_k)\in H^1(\mathbf{R}^3)^4$ for all $1\leq k\leq n$.