Dispersive estimates for 1D matrix Schrödinger operators with threshold resonance

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Motivation: asymptotic stability of 1D cubic NLS solitary waves

1D focusing cubic Schrödinger equation:

$$i\partial_t \psi + \partial_x^2 \psi + |\psi|^2 \psi = 0, \quad \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$$

Solitary wave solution:

$$e^{it} Q(x), \quad Q(x) = \sqrt{2}\,\mathrm{sech}(x)$$

Perturbation (no modulation):

$$\psi(t, x) = e^{it}(Q(x) + u(t, x))$$

Evolution equation for perturbation:

$$i\partial_t \begin{bmatrix} u \\ \bar{u} \end{bmatrix} - \mathcal{H} \begin{bmatrix} u \\ \bar{u} \end{bmatrix} = \begin{bmatrix} -Q(u^2 + 2u\bar{u}) \\ Q(\bar{u}^2 + 2u\bar{u}) \end{bmatrix} + \begin{bmatrix} -|u|^2 u \\ |u|^2 u \end{bmatrix}, \tag{1}$$

where

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} = \begin{bmatrix} -\partial_x^2 + 1 & 0 \\ 0 & \partial_x^2 - 1 \end{bmatrix} + \begin{bmatrix} -4\,\mathrm{sech}^2(x) & -2\,\mathrm{sech}^2(x) \\ 2\,\mathrm{sech}^2(x) & 4\,\mathrm{sech}^2(x) \end{bmatrix}$$

Goals:

- Study the spectrum of $\mathcal{H}$ and the linear flow $e^{it\mathcal{H}}$
- Analyze long-time behavior of small solutions of (1)

Outline

- Dispersive and local decay estimates on the line
- Previous works for the matrix case
- Spectrum of $\mathcal{H}$
- Characterizing threshold resonances
- Main result
- Application to the perturbation equation for 1D cubic NLS

Wave-type equations on the line

Let $a(\xi)$ be real. Consider

$$\begin{cases} i\partial_t u - a(\tfrac{1}{i}\partial_x)u = N(u,\bar{u}), & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0) = u_0 \end{cases} \tag{2}$$

Examples:

- $a(\xi) = \xi^2$ Schrödinger
- $a(\xi) = -\xi^3$ Airy (linear KdV)
- $a(\xi) = \langle \xi \rangle = (1+\xi^2)^{1/2}$ "half" Klein-Gordon
- $a(\xi) = \xi$ "half" wave (transport)

Linear solutions of (2):

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x\xi - ta(\xi))} \hat{u}_0(\xi) \mathrm{d}\xi$$

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resembles "oscillatory integral of the first kind"

$$\mathcal{I}(t) := \int_{\mathbb{R}} e^{it\phi(\xi)} g(\xi)\,\mathrm{d}\xi \tag{3}$$

$\rightsquigarrow$ Decay of $\mathcal{I}(t)$ depends on critical points of the phase $\phi(\xi)$ and the support/regularity of $g(\xi)$

Toolbox from Hamonic analysis

Lemma (Van der Corput)

Suppose $|\phi^{(k)}| \geq 1$ *and* $g, \partial_\xi g \in L^1(\mathbb{R})$. *Then,*

$$\left| \int_{\mathbb{R}} e^{it\phi(\xi)} g(\xi)\, \mathrm{d}\xi \right| \leq C_k t^{-\frac{1}{k}} \|\partial_\xi g\|_{L^1(\mathbb{R})} \tag{4}$$

holds when

- $k \geq 2$, *or*
- $k = 1$ *and* ϕ' *monotonic.*

Corollary

- *Schrödinger*

$$\|e^{it\partial_x^2} f\|_{L^\infty_x(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|\partial_\xi \hat{f}\|_{L^1_\xi(\mathbb{R})} \tag{5}$$

- *Airy (linear KdV)*

$$\|e^{t\partial_x^3} f\|_{L^\infty_x(\mathbb{R})} \lesssim |t|^{-\frac{1}{3}} \|\partial_\xi \hat{f}\|_{L^1_\xi(\mathbb{R})} \tag{6}$$

Improvement: use fundamental solution formulas and Young's convolution inequality to replace $\|\partial_\xi \hat{f}\|_{L^1_\xi(\mathbb{R})}$ by $\|f\|_{L^1_x(\mathbb{R})}$.

Lemma (Method of stationary phase)

Suppose $k \geq 2$, and

$$\phi(\xi_*) = \phi'(\xi_*) = \cdots = \phi^{(k-1)}(\xi_*) = 0, \tag{7}$$

while $\phi^{(k)}(\xi_*) \neq 0$. If g is supported in neighborhood of ξ_*, then

$$\int_{\mathbb{R}} e^{it\phi(\xi)} g(\xi)\,\mathrm{d}\xi \sim t^{-\frac{1}{k}} \left(a_0 + a_1 t^{-\frac{1}{k}} + a_2 t^{-\frac{2}{k}} + \cdots \right). \tag{8}$$

Linear Schrödinger equation:

$$u(t,x) = e^{it\partial_x^2} u_0(x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi it}} \hat{u}_0\left(\frac{x}{2t}\right) + r(x,t), \quad |r(x,t)| = \mathcal{O}(t^{-\frac{3}{2}})$$

$\rightsquigarrow$ Error term $r(x,t)$ can be precisely quantified by measuring L^p-norms on the moments: $\|\langle x\rangle^\alpha u_0\| \approx \|\langle \partial_\xi\rangle^\alpha \hat{u}_0\|$ or $\|\langle \partial_x\rangle^\alpha u_0\| \approx \|\langle \xi\rangle^\alpha \hat{u}_0\|$ for some $\alpha \geq 1$.

Dispersive estimates for Schödinger operators with potentials

Threshold resonance: there exists a non-trivial $\varphi \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$

$$H\varphi = \left(-\frac{d^2}{dx^2} + V(x)\right)\varphi = 0. \tag{9}$$

Remark: $H_0 = -\partial_x^2$ has a threshold resonance $\varphi(x) \equiv 1$.

Theorem (Dispersive estimates. [Weder '00], [Goldberg-Schlag '04])

With or without threshold resonance:

$$\|e^{itH} P_c f\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{1}{2}} \|f\|_{L^1(\mathbb{R})} \tag{10}$$

Theorem (Local decay estimates. [Goldberg '07])

Without threshold resonance:

$$\|\langle x\rangle^{-1} e^{itH} P_c f\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{3}{2}} \|\langle x\rangle f\|_{L^1(\mathbb{R})} \tag{11}$$

With threshold resonance:

$$\|\langle x\rangle^{-2}(e^{itH} P_c f - \tfrac{1}{\sqrt{-4\pi i t}}\langle \varphi, f\rangle \varphi)\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{3}{2}} \|\langle x\rangle^2 f\|_{L^1(\mathbb{R})} \tag{12}$$

Proof based on:

- distorted Fourier transform

$$\widetilde{f}(\xi) = \widetilde{\mathcal{F}}f(\xi) = \langle e(\cdot,\xi), f\rangle, \quad e(x,\xi) = \frac{1}{\sqrt{2\pi}} \begin{cases} T(\xi)f_+(x,\xi), & \xi \geq 0, \\ T(-\xi)f_-(x,\xi), & \xi < 0. \end{cases}$$

  where $T(\xi)$ is the transmission coefficient of the Jost solutions

$$Hf_\pm(x,\xi) = \xi^2 f_\pm(x,\xi), \quad f_\pm(x,\xi) \sim e^{\pm ix\xi} \text{ as } x \to \pm\infty.$$

- stationary phase analysis

$$e^{itH}P_c f = \int_{\mathbb{R}} e^{-it\xi^2}\widetilde{f}(\xi)e(x,\xi)d\xi.$$

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Question: Can we do a similar analysis for the 1D matrix Schrödinger operator

$$\mathcal{H} = \begin{bmatrix} -\partial_x^2 + \mu & 0 \\ 0 & \partial_x^2 - \mu \end{bmatrix} + \begin{bmatrix} -V_1(x) & -V_2(x) \\ V_2(x) & V_1(x) \end{bmatrix}.$$

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Answer: Known if $\mathcal{H}$ has no threshold resonance and no embedded eigenvalues, and $\mathcal{V}$ is even and exponentially decaying. [Buslaev-Perelman '95], [Krieger-Schlag '06]

Previous works: selected references

$$\mathcal{H} = \begin{bmatrix} -\Delta + \mu & 0 \\ 0 & \Delta - \mu \end{bmatrix} + \begin{bmatrix} -V_1(x) & -V_2(x) \\ V_2(x) & V_1(x) \end{bmatrix}$$

Spectral theory of $\mathcal{H}$:

- For $\mathcal{H}$ arising from NLS – [Weinstein '82-85], [Buslaev-Perelman '95], [Krieger-Schlag '06], [Chang et.al. '08]
- For general $\mathcal{H}$ and spectral representation of $e^{it\mathcal{H}}$ under reasonable assumptions (next slide) – [Erdogan-Schlag '06]

Dispersive estimates for $e^{it\mathcal{H}} P_{\mathrm{s}}$:

- 1D – (no threshold resonance) [Buslaev-Perelman '95], [Krieger-Schlag '06], [Collot-Germain '23]
- 2D – (no resonance/eigenvalue) [Green-Erdogan '13], (s-wave resonance) [Toprak '17]
- 3D – (no threshold resonance) [Schlag '04], (all scenarios) [Erdogan-Schlag '06]
- 5D – (no threshold eigenvalue) [Green '12]

Closely related:

- classification of threshold resonance for 1D scalar Schrödinger operator [Jensen-Nenciu '01], [Boussaid-Comech '22]
- dispersive estimates for 1D massive Dirac operator [Erdogan-Green '20]

Spectrum of $\mathcal{H}$ (non-self-adjoint operator)

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} = \begin{bmatrix} -\partial_x^2 + \mu & 0 \\ 0 & \partial_x^2 - \mu \end{bmatrix} + \begin{bmatrix} -V_1(x) & -V_2(x) \\ V_2(x) & V_1(x) \end{bmatrix}$$

Spectral assumptions on $\mathcal{H}$:

(A1) $-\sigma_3 \mathcal{V}$ is positive matrix, where σ_3 is the Pauli matrix.

(A2) $L_- := -\partial_x^2 + \mu - V_1 + V_2$ is non-negative,

(A3) there exists $\beta > 0$ such that $|V_1(x)| + |V_2(x)| \lesssim \langle x \rangle^{-\beta}$ for all $x \in \mathbb{R}$,

(A4) there are no embedded eigenvalues in $(-\infty, -\mu) \cup (\mu, \infty)$.

Lemma (Erdogan-Schlag, '06)

Assume (A1) – (A4). Then,

- $\sigma_{ess}(\mathcal{H}) = \sigma_{ess}(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty)$,
- $\mathrm{spec}(\mathcal{H}) = -\mathrm{spec}(\mathcal{H}) = \overline{\mathrm{spec}(\mathcal{H})} = \mathrm{spec}(\mathcal{H}^*) \subset \mathbb{R} \cup i\mathbb{R}$,
- *zero eigenvalues may have generalized eigenspaces, but finite dimensional,*
- *non-zero eigenvalues: algebraic and geometric multiplicities coincide and are finite.*

Characterizing regularity at spectrum endpoint $\lambda = \mu$

For $z \in \mathbb{C}_+$, let

$$\mathcal{R}(z) = (\mathcal{H} - (\mu + z^2))^{-1}, \quad \mathcal{R}_0(z) = (\mathcal{H}_0 - (\mu + z^2))^{-1}. \tag{13}$$

By (A1), write $\mathcal{V} = -\sigma_3 vv = v_1 v_2$. Symmetric resolvent identity:

$$\mathcal{R}(z) = \mathcal{R}_0(z) - \mathcal{R}_0 v_1 (I + v_2 \mathcal{R}_0(z) v_1)^{-1} v_2 \mathcal{R}_0(z) \tag{14}$$

Laurent expansion of free resolvent $\mathcal{R}_0(z)$:

$$\begin{bmatrix} \frac{ie^{iz|x-y|}}{2z} & 0 \\ 0 & \frac{e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}} \end{bmatrix} = \begin{bmatrix} \frac{i}{2z} & 0 \\ 0 & 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -\frac{|x-y|}{2} & 0 \\ 0 & -\frac{e^{-\sqrt{2\mu}|x-y|}}{2\sqrt{2\mu}} \end{bmatrix}}_{=:\mathcal{G}_0(x,y)} + \mathcal{O}(z|x-y|^2)$$

Definition (Regular)

Let T, P, Q be operators on $L^2(\mathbb{R})^2$ with integral kernels

$$T(x,y) = I + v_2(x)\mathcal{G}_0(x,y)v_1(y),$$
$$P(x,y) = \|V_1\|^{-1}_{L^1(\mathbb{R})} v_2(x)\underline{e}_{11} v_1(y), \quad Q = I - P.$$

We say μ is regular if QTQ is invertible on the subspace $Q(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$

$\rightsquigarrow$ closely related to Birman-Schwinger principle.

Lemma (Characterizing threshold resonances of $\mathcal{H}$)

Suppose μ is not regular (A5), and assumptions (A1) – (A4) hold.

1. *Let $0 \neq \vec{\Phi} \in \ker(QTQ)$. Then $\vec{\Psi} := v_2^{-1}\vec{\Phi} \in L^\infty(\mathbb{R})^2$ is a distributional solution to $\mathcal{H}\vec{\Psi} = \mu\vec{\Psi}$. Furthermore, if*

$$c_{2,\pm} := \int_{\mathbb{R}} e^{\pm\sqrt{2\mu}y}(V_2\Psi_1 + V_1\Psi_2)(y)dy = 0 \qquad \text{(A6)}$$

   then $\Psi_1 \notin L^2(\mathbb{R})$ and $\dim(\ker(QTQ)) = 1$.

2. *Conversely, if there exists $0 \neq \vec{\Psi} \in L^\infty(\mathbb{R})^2$ satisfying $\mathcal{H}\vec{\Psi} = \mu\vec{\Psi}$, then $\vec{\Phi} = v_2\vec{\Psi} \in \ker(QTQ)$.*

Remark: (A6) condition verified for the linearized operator (of 1D cubic NLS)

$$\mathcal{H}_1 = \begin{bmatrix} -\partial_x^2 + 1 & 0 \\ 0 & \partial_x^2 - 1 \end{bmatrix} + \begin{bmatrix} -4\operatorname{sech}^2(x) & -2\operatorname{sech}^2(x) \\ 2\operatorname{sech}^2(x) & 4\operatorname{sech}^2(x) \end{bmatrix}, \quad \vec{\Psi}(x) = \begin{bmatrix} \tanh^2(x) \\ -\operatorname{sech}^2(x) \end{bmatrix}$$

Verification:

$$\int_{\mathbb{R}} e^{sy}(4\operatorname{sech}^2(y)\tanh^2(y) - 2\operatorname{sech}^4(y))dy = \frac{\pi s(s^2-2)}{\sin(\frac{\pi s}{2})} \implies c_{2,\pm} = 0$$

Main result

Theorem (L.)

Suppose (A1) – (A6) hold, and let $\vec{\Psi} = (\Psi_1, \Psi_2) \in L^\infty \setminus L^2$ *be the distributional solution to*

$$\mathcal{H}\vec{\Psi} = \mu\vec{\Psi},$$

with the normalization

$$\lim_{x\to\infty} \left(|\Psi_1(x)|^2 + |\Psi_1(-x)|^2 \right) = 2.$$

Then, for any $\vec{f} = (f_1, f_2) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ *and* $|t| \geq 1$*, we have*

1. *the dispersive estimate*

$$\left\| e^{it\mathcal{H}} P_{\mathrm{s}} \vec{f} \right\|_{L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \left\| \vec{f} \right\|_{L^1(\mathbb{R})\times L^1(\mathbb{R})},$$

2. *and local decay estimate*

$$\left\| \langle x\rangle^{-2} (e^{it\mathcal{H}} P_{\mathrm{s}} - F_t)\vec{f} \right\|_{L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \left\| \langle x\rangle^{2} \vec{f} \right\|_{L^1(\mathbb{R})\times L^1(\mathbb{R})},$$

   where

$$F_t\vec{f} := \frac{e^{it\mu}}{\sqrt{-4\pi it}} \langle \sigma_3\vec{\Psi}, \vec{f}\,\rangle\vec{\Psi} - \frac{e^{-it\mu}}{\sqrt{4\pi it}} \langle \sigma_3\sigma_1\vec{\Psi}, \vec{f}\,\rangle\sigma_1\vec{\Psi}.$$

Short proof summary (small energies)

Spectral representation

$$e^{it\mathcal{H}}P_{\mathrm{s}}^{+} = \frac{e^{it\mu}}{\pi i}\int_{\mathbb{R}} e^{itz^2} z\mathcal{R}(z)dz,$$

where

$$\begin{aligned}\mathcal{R}(z) &= \mathcal{R}_0(z) - \mathcal{R}_0(z)v_1(M(z))^{-1}v_2\mathcal{R}_0(z),\\ M(z) &= I + v_2\mathcal{R}_0(z)v_1.\end{aligned}$$

Then, for $0 < |z| < z_0$ and z_0 small,

$$M(z)^{-1} = \frac{c_0}{z}S_1 + c_1 PTS_1 + c_2 S_1 TP + c_3 zP + \mathcal{O}(z^2)$$

with $P = v_2\underline{e}_{11}v_1$ and $S_1 = (v_2\vec{\Psi})\otimes(v_2\vec{\Psi})$.
$\Longrightarrow$ Use oscillatory integral tools to prove dispersive estimates

$$\|e^{it\mathcal{H}}\chi_0(\mathcal{H}-\mu I)P_{\mathrm{s}}^{+}\vec{u}\|_{L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}}\|\vec{u}\|_{L^1(\mathbb{R})\times L^1(\mathbb{R})}$$

$$\|\langle x\rangle^{-2}(e^{it\mathcal{H}}\chi_0(\mathcal{H}-\mu I)P_{\mathrm{s}}^{+} - F_t^{+})\vec{u}\|_{L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}}\|\langle x\rangle^2\vec{u}\|_{L^1(\mathbb{R})\times L^1(\mathbb{R})}$$

where $F_t^{+} = \frac{e^{it\mu}}{\sqrt{-4\pi it}}\vec{\Psi}\otimes\sigma_3\vec{\Psi}$.

Application: 1D cubic NLS – perturbation equation (without modulation)

Recall $Q(x) = \sqrt{2}\,\mathrm{sech}(x)$ and (1):

$$i\partial_t \begin{bmatrix} u \\ \bar{u} \end{bmatrix} - \mathcal{H}_1 \begin{bmatrix} u \\ \bar{u} \end{bmatrix} = \underbrace{\begin{bmatrix} -Q(u^2 + 2u\bar{u}) \\ Q(\bar{u}^2 + 2u\bar{u}) \end{bmatrix}}_{=:\mathcal{Q}(U)} + \begin{bmatrix} -|u|^2 u \\ |u|^2 u \end{bmatrix}, \quad U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$$

Threshold resonance:

$$\mathcal{H}_1 = \begin{bmatrix} -\partial_x^2 - 4\,\mathrm{sech}^2(x) + 1 & -2\,\mathrm{sech}^2(x) \\ 2\,\mathrm{sech}^2(x) & \partial_x^2 + 4\,\mathrm{sech}^2(x) - 1 \end{bmatrix}, \quad \vec{\Psi}(x) = \begin{bmatrix} \tanh^2(x) \\ -\mathrm{sech}^2(x) \end{bmatrix}.$$

Fix $F \in \mathcal{S}(\mathbb{R})^2$ and consider

$$U_{\mathrm{free}}(t) := e^{-it\mathcal{H}_1} P_{\mathrm{s}} F. \tag{15}$$

Leading order decomposition (via Theorem):

$$U_{\mathrm{free}}(t,x) = c_- \frac{e^{-it}}{\sqrt{t}} \Psi(x) + c_+ \frac{e^{it}}{\sqrt{t}} \sigma_1 \Psi(x) + R(t,x), \quad \|\langle x \rangle^{-2} R(t,x)\|_{L^\infty} \lesssim |t|^{-\frac{3}{2}}.$$

Localized quadratic nonlinear term:

$$\mathcal{Q}(U_{\mathrm{free}}) = c_+^2 \frac{e^{2it}}{t} \mathcal{Q}_1(\Psi) + c_+ c_- \frac{1}{t} \mathcal{Q}_2(\Psi) + c_-^2 \frac{e^{-2it}}{t} \mathcal{Q}_3(\Psi) + \mathcal{O}_{L^\infty}(t^{-2})$$

Examining source term contributed by threshold resonance...

Inhomogeneous linear equation:

$$\begin{cases} i\partial_t V - \mathcal{H}_1 V = P_{\mathrm{s}}\left(c_+^2 \frac{e^{2it}}{t}\mathcal{Q}_1(\Psi) + c_+ c_- \frac{1}{t}\mathcal{Q}_2(\Psi) + c_-^2 \frac{e^{-2it}}{t}\mathcal{Q}_3(\Psi)\right) \\ V(1) = (0,0) \end{cases}$$

Darboux transformation [Martel '21]: there is an (explicit) operator $\mathcal{D}$ such that

$$\mathcal{D}\mathcal{H}_1 = \mathcal{H}_0\mathcal{D}, \quad \text{where } \mathcal{H}_0 := \begin{bmatrix} -\partial_x^2 + 1 & 0 \\ 0 & \partial_x^2 - 1 \end{bmatrix}$$

Equation for transformed variable $W := \mathcal{D}V$

$$i\partial_t W - \mathcal{H}_0 W = \mathcal{D}\left(c_+^2 \frac{e^{2it}}{t}\mathcal{Q}_1(\Psi) + c_+ c_- \frac{1}{t}\mathcal{Q}_2(\Psi) + c_-^2 \frac{e^{-2it}}{t}\mathcal{Q}_3(\Psi)\right)$$

Duhamel formulation

$$W(t) = -ie^{-it\mathcal{H}_0}\int_1^t e^{is\mathcal{H}_0}\mathcal{D}\left(c_+^2 \frac{e^{2is}}{s}\mathcal{Q}_1(\Psi) + c_+ c_- \frac{1}{s}\mathcal{Q}_2(\Psi) + c_-^2 \frac{e^{-2is}}{s}\mathcal{Q}_3(\Psi)\right) ds$$

Representation of $e^{it\mathcal{H}_0}$ by Fourier transform

$$(2\pi)^{\frac{1}{2}} e^{-it\mathcal{H}_0}\vec{f}(x) = \int_{\mathbb{R}} e^{i(x\xi - t(\xi^2+1))}\hat{f}_1(\xi)d\xi \underline{e}_1 + \int_{\mathbb{R}} e^{i(x\xi + t(\xi^2+1))}\hat{f}_2(\xi)d\xi \underline{e}_2$$

Null structure for quadratic nonlinearity

Fourier transform of quadratic source term: $\mathcal{F}[e^{it\mathcal{H}_0}W(t)](\xi) =$

$$\int_1^t \frac{e^{is(\xi^2+3)}}{s}\widehat{G_{1,1}}(\xi)ds + \int_1^t \frac{e^{is(\xi^2+1)}}{s}\widehat{G_{2,1}}(\xi)ds + \int_1^t \frac{e^{is(\xi^2-1)}}{s}\widehat{G_{3,1}}(\xi)ds$$

$$+ \int_1^t \frac{e^{-is(\xi^2-1)}}{s}\widehat{G_{1,2}}(\xi)ds + \int_1^t \frac{e^{-is(\xi^2+1)}}{s}\widehat{G_{2,2}}(\xi)ds + \int_1^t \frac{e^{-is(\xi^2+3)}}{s}\widehat{G_{3,2}}(\xi)ds$$

where $G_{j,k} = (\mathcal{D}\mathcal{Q}_j(\Psi))_k$. Time resonant (bad) frequencies at $\xi = \pm 1$ however

$$\widehat{G_{3,1}}(\pm 1) = \widehat{G_{1,2}}(\pm 1) = 0. \tag{16}$$

Proof (Mathematica-assisted):

$$G_{3,1}(x) = G_{1,2}(x) = 250\,\mathrm{sech}^3(x) - 3720\,\mathrm{sech}^5(x) + 9960\,\mathrm{sech}^7(x) - 6720\,\mathrm{sech}^9(x)$$

$$\Longrightarrow \widehat{G_{3,1}}(\xi) = \widehat{G_{1,2}}(\xi) = -\frac{\sqrt{\pi}}{6\sqrt{2}}(\xi^2 - 1)\xi^2(\xi^2+1)^2\,\mathrm{sech}(\tfrac{\pi\xi}{2}).$$

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$$+ \int_1^t \frac{e^{-is(\xi^2-1)}}{s}\widehat{G_{1,2}}(\xi)ds + \int_1^t \frac{e^{-is(\xi^2+1)}}{s}\widehat{G_{2,2}}(\xi)ds + \int_1^t \frac{e^{-is(\xi^2+3)}}{s}\widehat{G_{3,2}}(\xi)ds$$

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$$\widehat{G_{3,1}}(\pm 1) = \widehat{G_{1,2}}(\pm 1) = 0. \tag{16}$$

Proof (Mathematica-assisted):

$$G_{3,1}(x) = G_{1,2}(x) = 250\,\mathrm{sech}^3(x) - 3720\,\mathrm{sech}^5(x) + 9960\,\mathrm{sech}^7(x) - 6720\,\mathrm{sech}^9(x)$$
$$\Longrightarrow \widehat{G_{3,1}}(\xi) = \widehat{G_{1,2}}(\xi) = -\frac{\sqrt{\pi}}{6\sqrt{2}}(\xi^2-1)\xi^2(\xi^2+1)^2\,\mathrm{sech}(\tfrac{\pi\xi}{2}).$$

Open problem: give a perturbative proof of asymptotic stability for NLS solitary wave

$$i\partial_t\psi + \partial_x^2\psi + |\psi|^{p-1}\psi = 0, \quad 1 < p < 5.$$

Thank you for your attention!