Strichartz estimates for the Schrödinger equation on negatively curved compact manifolds.

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Setting

Compact boundaryless Riemannian manifold (M^d, g) of dimension $d \geq 2$.

The associated Laplace-Beltrami Operator, Δ_g, of the form

$$|g|^{1/2} \sum_{j,k=1}^{d} \partial_j(|g|^{1/2} g^{jk}(x))\partial_k, \quad |g| = det\, g_{jk}(x),\, (g^{jk}) = (g_{jk})^{-1}.$$

The Laplace-Beltrami Operator is self–adjoint. Additionally, the spectrum of $-\Delta_g$ is discrete. We often denote $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \dots$ for the ordered sequence of eigenvalues, repeated according to multiplicity.

Setting

Let

$$u(x,t) = \left(e^{-it\Delta_g} u_0\right)(x)$$

be the solution of the Schrödinger equation on $M^d \times \mathbb{R}$,

$$i\partial_t u(x,t) = \Delta_g u(x,t), \quad u(x,0) = u_0(x).$$

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- Interested in the space time estimate $\|u\|_{L_t^p L_x^q(M^d \times [0,1])}$, where

$$\|u\|_{L_t^p L_x^q(M^d \times [0,1])} = \Big(\int_0^1 \|u(\cdot, t)\|_{L_x^q(M^d)}^p \, dt \Big)^{1/p}.$$

The Euclidean case

In $\mathbb{R}^d$,

$$u(x) = \left(e^{-it\Delta}u_0\right)(x) = \int e^{2\pi i\langle x-y,\xi\rangle} e^{it|\xi|^2} \widehat{u_0}(\xi) d\xi$$
$$= \frac{1}{(4\pi it)^{d/2}} \int e^{-i\frac{|x-y|^2}{4t}} u_0(y) dy$$

We have the mixed-norm Strichartz estimates

$$\|u\|_{L_t^p L_x^q(\mathbb{R}^d\times\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}$$

for all *admissible* pairs (p, q) satisfying

$$d(\tfrac{1}{2} - \tfrac{1}{q}) = \tfrac{2}{p} \text{ and } 2 < q \leq \tfrac{2d}{d-2} \text{ if } d \geq 3, \text{ or } 2 < q < \infty \text{ if } d = 2.$$

Strichartz 1977, Ginibre–Velo 1992, Keel–Tao 1998

Dispersive estimate

The Strichartz estimate in $\mathbb{R}^d$ follows from

$$\|u(\cdot,t)\|_{L^\infty} \le \frac{C}{t^{d/2}}\|u_0\|_{L^1}, \text{ the dispersive estimate}$$

as well as

$$\|u(\cdot,t)\|_{L^2} = \|u_0\|_{L^2}$$

• On compact manifolds we do not have the dispersive estimate, even for small time t, e.g., by letting $u_0 = e_\lambda$ with $\lambda \to \infty$.

Strichartz estimate on compact manifold

Burq, Gérard and Tzvetkov 2004: Let M^d be a $d \geq 2$ dimensional compact manifold. Then for all *admissible* pairs (p, q),

$$\|u\|_{L_t^p L_x^q(M^d \times [0,1])} \lesssim \|u_0\|_{H^{1/p}(M^d)}.$$

- The *admissible* pairs (p, q) are the same as in the Euclidean space
- H^μ denotes the standard Sobolev space

$$\|f\|_{H^\mu(M^d)} = \|(I+P)^\mu f\|_{L^2(M^d)}, \quad \text{with} \quad P = \sqrt{-\Delta_g},$$

• On compact manifold, one can not replace $[0, 1]$ by $\mathbb{R}$ by letting

$$u(x, t) = e^{it\lambda^2} e_\lambda(x)$$

where

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda,$$

is an eigenfunction of the the Laplacian.

• For the endpoint Strichartz estimates where $p = 2$ and $q = \frac{2d}{d-2}$ with $d \geq 3$, the $1/2$ derivative loss in the Strichartz estimate is sharp by letting $u = Z_\lambda$, the zonal eigenfunctions on S^d with eigenvalue $\lambda = (k(k + d - 1))^{1/2}$, $k = 1, 2, \ldots$, since

$$\|Z_\lambda\|_{L^{\frac{2d}{d-2}}(S^d)} / \|Z_\lambda\|_{L^2(S^d)}. \approx \lambda^{1/2}$$

- Can we improve the Strichartz estimate for non-endpoint pairs (p, q)?

- For the endpoint pair $p = 2$ and $q = \frac{2d}{d-2}$, can we get an improvement under certain geometric assumptions?

Improved Strichartz estimates

Theorem 1 (Huang–Sogge 2024).

Let M^d be a $d \geq 2$ dimensional compact manifold all of whose sectional curvatures are nonpositive. Then for all admissible *pairs* (p, q),

$$\|u\|_{L_t^p L_x^q(M^d \times [0,1])} \lesssim \left\|(I+P)^{1/p} \left(\log(2I+P)\right)^{-\frac{1}{p}} f\right\|_{L^2(M^d)}.$$

Main ideas

The results of Burq, Gérard and Tzvetkov 2004 from the uniform bounds

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L^p_t L^q_x(M^d\times[0,1])} \leq C\lambda^{\frac{1}{p}}\,\|f\|_{L^2(M^d)}, \quad \lambda \gg 1.$$

The authors proved this estimate by showing that one always has the following uniform dyadic estimates over very small intervals:

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L^p_t L^q_x(M^d\times[0,\lambda^{-1}])} \leq C\,\|f\|_{L^2(M^d)}, \quad \lambda \gg 1.$$

• This implies the above estimates by diving the interval [0,1] into $\approx \lambda$ many smaller intervals of size λ^{-1}, $[j\lambda^{-1},(j+1)\lambda^{-1}]$, and use the fact that $e^{-i\cdot j\lambda^{-1}\Delta_g}$ is unitary on L^2.

Main ideas

When $t \leq \lambda^{-1}$, Burq, Gérard and Tzvetkov showed the following dispersive type estimate

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L^\infty_x(M^d)} \leq C|t|^{-d/2}\|f\|_{L^1(M^d)}, \quad |t| \leq \lambda^{-1}.$$

• This was proved using parametrix construction, when $t \leq \lambda^{-1}$, $e^{-it\Delta_g}\beta(P/\lambda)$ behaves like the half wave operator $e^{it'\sqrt{-\Delta_g}}\beta(P/\lambda)$ with $t' \leq 1$

Main ideas

If $t' = \lambda t \lesssim 1$, by using WKB approximation or Hadamard parametrix, we have

$$e^{it'\lambda^{-1}\Delta_g}(x,y) \approx \int e^{i\langle x-y,\xi\rangle} e^{it'\lambda^{-1}|\xi|^2} \beta(|\xi|/\lambda) d\xi$$

- This implies the decay estimates at scale $t' \lesssim 1$ by stationary phase argument.

Main ideas

In an ongoing work with Sogge, for manifolds with non-positive curvature, we obtain

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L_t^pL_x^q(M^d\times[0,\lambda^{-1}\log\lambda])}\leq C\,\|f\|_{L^2(M^d)},\quad \lambda\gg 1.$$

- This implies the results in the theorem by adding up the disjoint intervals.
- Unlike the general case, we do not have analogous dispersive estimate at scale $t\leq\lambda^{-1}\log\lambda$

Main ideas

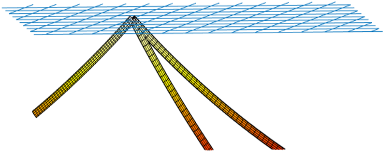
$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L^p_t L^q_x(M^d\times[0,\lambda^{-1}\log\lambda])}$$
$$\lesssim \|A^\theta_\nu e^{-it\Delta_g}\beta(P/\lambda)f\|_{L^p_t \ell^q_\nu L^q_x(M^d\times[0,\lambda^{-1}\log\lambda])} + \text{good error}.$$

• The A^θ_ν operators involves localizations in both space and frequency, as an analog of the wave packet in $\mathbb{R}^d$.

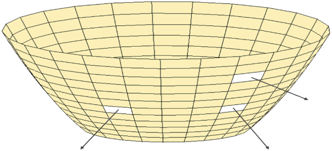
• If one view $\ell^q_\nu L^q$ norm as L^q norm on some Banach space X, the dispersive estimate holds on the new space X, due to the extra directional localization.

• This is proved using bilinear oscillatory integral estimate of Lee 2006.

Main ideas



The Schrödinger tubes



The Euclidean case

The flat tori

On the torus, or more generally flat manifold,

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L_t^p L_x^q(M^d\times[0,\lambda^{-1+\delta_d}])} \le C\,\|f\|_{L^2(M^d)}, \quad \lambda \gg 1,$$

for some small δ_d depend on the dimension.

For the standard torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $q = \frac{2(d+2)}{d}$, Bourgain Demeter 2014.

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L_t^q L_x^q(\mathbb{T}^d\times[0,1])} \le C_\varepsilon \lambda^\varepsilon\,\|f\|_{L^2(\mathbb{T}^d)}, \quad \text{for arbitrary small } \varepsilon.$$

• When $d = 2$, λ^ε loss is improved to $(\log\lambda)^{\frac{1}{4}}$ recently by Herr Kwak 2023.

• For irrational tori, [0,1] can be replaced by [0, T] and the dependence of C_ε on the constant T has been studied Deng, Germain Guth 2017.

The standard round sphere

On the round sphere S^d, it is possible to improve the Strichartz estimate for non-endpoint pairs (p, q) using the arithmetic properties of the spectrum.
Burq, Gérard and Tzvetkov 2004

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L_t^4 L_x^4(S^d\times[0,1])} \leq C\lambda^{s(d)}\|f\|_{L^2(S^d)}, \quad \lambda \gg 1,$$

$s(d) = \frac{d}{4} - \frac{1}{2}$ for $d \geq 3$, and $s(d) = \frac{1}{8}$, for $d = 2$.

• This is sharp in all dimensions by using spherical harmonics as test functions.

• The result also holds on Zoll manifolds, which share similar spectrum properties with S^d.

The standard round sphere

On the round sphere S^d,

$$e^{-it\Delta_g}\beta(P/\lambda)f = \sum_k e^{itk(k+d-1)}\beta(|k|/\lambda)e_k(x),$$

Thus the Strichartz estimate is related to

$$\| \sum_{j\in\mathbb{Z}:|j|\approx\lambda} a_j e^{i\cdot j^2\theta}\|_{L^q(S^1)} \leq C_\varepsilon \lambda^{2(\frac{1}{4}-\frac{1}{q})+\varepsilon}\|a_j\|_{\ell^2}.$$

- For $q = 4$, this is related to the lattice points on a circle, i.e., $j_1^2 + j_2^2 = \lambda^2$.

Thank you for your time!