Virtual levels and virtual states of linear operators in Banach spaces

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The limiting absorption principle

Let $\mathcal{H}$ be a Hilbert space and A a self-adjoint operator on $\mathcal{H}$

$$\sigma(A) \subset \mathbb{R}.$$

Let **E** and **F** be Banach spaces such that:

$$\mathbf{E} \overset{\imath}{\hookrightarrow} \mathcal{H} \overset{\jmath}{\hookrightarrow} \mathbf{F}.$$

The limiting absorption principle (LAP) for A holds at $\lambda \in \sigma(A)$ means that

$$\underset{\epsilon \to 0+}{\text{w-lim}} (A - \lambda - \mathrm{i}\epsilon)^{-1}$$

exists in (weak operator topology of) $\mathscr{B}(\mathbf{E}, \mathbf{F})$.

LAP and spectrum

Let $\lambda \in \sigma(A)$. If $\phi \in \mathbf{E}$ is such that $A\phi = \lambda\phi$ then for $\epsilon > 0$

$$(A - \lambda - \mathrm{i}\epsilon)^{-1}\phi = \frac{\mathrm{i}}{\epsilon}\phi.$$

If the LAP holds for A at λ then $\phi = 0$.

Assume LAP holds for A at λ. Let $B \in \mathscr{B}(\mathbf{F}, \mathbf{E})$ then if B is small enough

$$I_{\mathbf{E}} + B(A - \lambda - \mathrm{i}\epsilon)^{-1}$$

is invertible, for $\epsilon > 0$ small enough (and also at the limit $\epsilon = 0$). The inverse has a limit as $\epsilon \to 0+$. Therefore LAP holds for $A + B$ at λ and $A + B$ has no eigenvector associated with λ.

LAP and evolution

Consider $\mathbf{F} = \mathbf{E}^*$. Assume LAP holds for A at any point of an interval I and $C \in \mathscr{B}(\mathbf{F}, \mathcal{H})$ then

$$\int_I \|C\Im(A-\lambda-\mathrm{i}\epsilon)^{-1}\phi\|^2 \, \mathrm{d}\lambda \leq \kappa\|\phi\|^2$$

where κ does not depend on $\epsilon > 0$. By means of a Fourier transform in λ, it leads to

$$\int_{\mathbb{R}} \|Ce^{-\mathrm{i}tA}\mathbb{1}_I(A)\phi\|^2 \, \mathrm{d}t \leq \kappa\|\phi\|^2.$$

LAP equivalent charaterization in the Schrödinger case

Consider the Schrödinger operator $A = -\Delta + V$, in $\mathbb{R}^d$, $d \geq 1$, with $V \in C_{\mathrm{comp}}(\mathbb{R}^d)$.

The following properties seem related for $z_0 = 0$:

1. The equation $A\psi = z_0\psi$ has a nonzero solution from L^2 or from a certain larger space;

2. The resolvent $R(z) = (A - zI)^{-1}$ has no limit in weighted spaces as $z \to z_0$;

3. Under some arbitrarily small perturbation, an eigenvalue can bifurcate from z_0.

In this case $z_0 = 0$ is called a threshold resonance.

Thresholds resonances $\rightsquigarrow$ virtual levels

Let $\mathbf{X}$ an infinite-dimensional complex Banach space.
Let $A \in \mathscr{C}(\mathbf{X})$ a closed linear operator with dense domain $\mathrm{D}(A)$.
Consider continuous embeddings (not necessarily dense)

$$\mathbf{E} \overset{\imath}{\hookrightarrow} \mathbf{X} \overset{\jmath}{\hookrightarrow} \mathbf{F}.$$

Let $\Omega \subset \mathbb{C} \setminus \sigma(A)$. Then $z_0 \in \sigma_{\mathrm{ess}}(A) \cap \partial\Omega$ is a *point of the essential spectrum of rank* $r \in \mathbb{N}_0$ *relative to* $(\Omega, \mathbf{E}, \mathbf{F})$ if $r \in \mathbb{N}_0$ is the smallest integer with

- B of rank r;
- $\Omega \cap \sigma(A + B) \cap \mathbb{D}_\delta(z_0) = \emptyset$, $\delta > 0$;
- In $\mathscr{B}(\mathbf{E}, \mathbf{F})$

$$(A + B - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}} := \underset{z \to z_0,\, z \in \Omega \cap \mathbb{D}_\delta(z_0)}{\text{w-lim}} \jmath \circ (A + B - z I_{\mathbf{X}})^{-1} \circ \imath.$$

If $r > 0$, z_0 is a virtual level.

Main assumption

The operator A, considered as a mapping $\mathbf{F} \to \mathbf{F}$,

$$\mathrm{D}(A_{\mathbf{F}\to\mathbf{F}}) := \jmath(\mathrm{D}(A)), \qquad A_{\mathbf{F}\to\mathbf{F}} : \Psi \mapsto \jmath(A\jmath^{-1}(\Psi)),$$

is closable in $\mathbf{F}$ $\rightsquigarrow$ closure $\hat{A} \in \mathscr{C}(\mathbf{F})$.

Lemma

If LAP holds at z_0 ($r = 0$) relative to $(\Omega, \mathbf{E}, \mathbf{F})$, then the limit $(A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}}$ is invertible with closed inverse α such that

$$\jmath \circ \imath \circ \alpha = (\hat{A} - z_0 I_{\mathbf{F}}) \restriction_{\mathrm{Ran}((A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}})} .$$

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If LAP does not hold (z_0 of rank $r > 0$), does it mean that the equation $(\hat{A} - z_0)\psi = 0$ has a nontrivial solution $\psi \in \mathbf{F}$?

Some notations

Let $\mathcal{B} \in \mathscr{C}(\mathbf{F}, \mathbf{E})$ a closed operator and

$$B := \imath \circ \mathcal{B} \circ \jmath \in \mathscr{C}(\mathbf{X}) \quad \text{and} \quad \hat{B} := \jmath \circ \imath \circ \mathcal{B} \in \mathscr{C}(\mathbf{F}).$$

If $\mathcal{B} \in \mathscr{B}_0(\mathrm{D}(\hat{A}), \mathbf{E})$ ($\hat{A}$-compact) then

$$B \in \mathscr{B}_0(\mathrm{D}(A), \mathbf{X}) \quad \text{and} \quad \hat{B} \in \mathscr{B}_0(\mathrm{D}(\hat{A}), \mathbf{F}).$$

Moreover, if A satisfies the main assumption then $A + B$ also satisfies the main assumption (closure is $\hat{A} + \hat{B}$).

Beside A satisfies the main assumption, assume LAP at z_0 holds for A w.r.t. to $(\Omega, \mathbf{E}, \mathbf{F})$.

Let $\mathcal{B} \in \mathscr{B}_0(\mathrm{D}(\hat{A}), \mathbf{E})$, does LAP at z_0 holds for $A + B$ w.r.t. to $(\Omega, \mathbf{E}, \mathbf{F})$?

Equivalent characterizations of virtual levels

Theorem

The following statements are equivalent:

1. *There is no nonzero solution to* $(\hat{A} + \hat{B} - z_0 I_{\mathbf{F}})\Psi = 0$, $\Psi \in \mathrm{Ran}\big((A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}}\big)$;
2. $-1 \notin \sigma\big(\mathcal{B}(A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}}\big)$;
3. $-1 \notin \sigma_{\mathrm{p}}\big((A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}}\mathcal{B}\big)$;
4. *There is* $\delta > 0$ *such that* $\Omega \cap \mathbb{D}_\delta(z_0) \subset \mathbb{C} \setminus \sigma(A + B)$, *and there exists a limit in* $\mathscr{B}(\mathbf{E}, \mathbf{F})$

   $$(A + B - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}} = \operatorname*{w-lim}_{z \to z_0,\, z \in \Omega \cap \mathbb{D}_\delta(z_0)} \jmath \circ (A + B - z I_{\mathbf{X}})^{-1} \circ \imath;$$

5. *There is* $\delta > 0$ *such that* $\Omega \cap \mathbb{D}_\delta(z_0) \subset \mathbb{C} \setminus \sigma(A + B)$, *and the mapping* $\jmath \circ (A + B - z I_{\mathbf{X}})^{-1} \circ \imath : \mathbf{E} \to \mathbf{F}$ *is uniformly bounded in* $z \in \Omega \cap \mathbb{D}_\delta(z_0)$.

Sketched proof

These equivalences are based on properties of Fredholm operators.

1, 2, and 3 are equivalent due to

$$\begin{aligned}\hat{A} + \hat{B} - z_0 I_{\mathbf{F}} &= \left(I_{\mathbf{F}} + \mathcal{B}(A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}} \right) \left(\hat{A} - z_0 I_{\mathbf{F}} \right) \\ &= \left(\hat{A} - z_0 I_{\mathbf{F}} \right) \left(I_{\mathbf{E}} + (A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}} \mathcal{B} \right)\end{aligned}$$

2 is equivalent to 4 (and 5) follows from the identity for $z \in \Omega$

$$A + B - zI_{\mathbf{X}} = \left(I_{\mathbf{X}} + B(A - zI_{\mathbf{X}})^{-1} \right) (A - zI_{\mathbf{X}})$$

so that it is enough to have that $\left(I_{\mathbf{X}} + B(A - zI_{\mathbf{X}})^{-1} \right)$ is invertible and the inverse has a limit as $z \to z_0$ iff $-1 \notin \sigma\left(\mathcal{B}(A - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}} \right)$.

Relatively compact instead of finite rank perturbations

Assume z_0 is a virtual level of A (of rank $r \geq 1$) relative to $(\Omega, \mathbf{E}, \mathbf{F})$.

Definition (Relatively compact regularizing operators)

$\mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}}) :=$

$$\left\{ \begin{array}{c} \mathcal{B} : \mathbf{F} \to \mathbf{E} \text{ is } \hat{A}\text{-compact}, \qquad \Omega \subset \mathbb{C} \setminus \sigma_{\mathrm{ess}}(A+B), \\ \exists \underset{z \to z_0,\, z \in \Omega}{\mathrm{w\text{-}lim}} \jmath \circ (A + B - z I_{\mathbf{X}})^{-1} \circ \imath : \mathbf{E} \to \mathbf{F} \end{array} \right\}.$$

Compact to finite rank operators

Lemma

For any $\mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$ *and the Riesz projection* $P_1 \in \mathscr{B}_{00}(\mathbf{E})$ *(finite rank) onto the generalized eigenspace corresponding to eigenvalue* $\lambda = 1$ *of the operator*

$$K = \mathcal{B}(A + B - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}} \in \mathscr{B}_0(\mathbf{E}),$$

there is the inclusion

$$P_1 \circ \mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}}).$$

Lemma

Assume that $\mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$ *and* $\operatorname{rank}\mathcal{B} < \infty$. *For any projection* $Q \in \mathscr{B}_{00}(\mathbf{F})$ *onto* $(A + B - z_0 I)^{-1}_{\Omega,\mathbf{E},\mathbf{F}} \operatorname{Ran}(\mathcal{B}) \subset \mathbf{F}$, *one has:*

$$\mathcal{B} \circ Q \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}}) \cap \mathscr{B}_{00}(\mathbf{F}, \mathbf{E}).$$

$\mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$ is “dense” and “open”

Lemma

If $\mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$, then there is $\epsilon > 0$ such that for all $\zeta \in \mathbb{D}_\epsilon \setminus \{0\}$ LAP holds at z_0 for $A + \zeta B$, relative to $(\Omega, \mathbf{E}, \mathbf{F})$.

Lemma

If $\mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$ and $\mathcal{C} : \mathbf{F} \to \mathbf{E}$ is $\hat{A}$-compact with

$$\sup_{z \in \Omega \cap \mathbb{D}_\delta(z_0)} \|\mathcal{C} \circ \jmath \circ (A + B - zI_{\mathbf{X}})^{-1} \circ \imath\|_{\mathbf{E} \to \mathbf{E}} < 1,$$

with some $\delta > 0$, then

$$\Omega \cap \mathbb{D}_\delta(z_0) \subset \mathbb{C} \setminus \sigma(A + B + C),$$

with $C := \imath \circ \mathcal{C} \circ \jmath : \mathbf{X} \to \mathbf{X}$, and LAP holds at z_0 for $A + B + C$, relative to $(\Omega, \mathbf{E}, \mathbf{F})$.

The space of virtual states

The space of virtual states $\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$ is

$$\left\{\Psi \in \mathrm{Ran}\big((A + B - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}}\big),\ (\hat{A} - z_0 I_{\mathbf{F}})\Psi = 0\right\}$$

where $\mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$.

Theorem

The spaces $\mathrm{Ran}\big((A + B - z_0 I_{\mathbf{X}})^{-1}_{\Omega,\mathbf{E},\mathbf{F}}\big)$ *and* $\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$ *do not depend on the choice of* $\mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})$*;*

Let $r := \min\{\mathrm{rank}\mathcal{B},\ \mathcal{B} \in \mathscr{Q}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}})\} < \infty$*, one has*

$$\dim \mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I_{\mathbf{X}}) = r.$$

Schrödinger in dimension $d = 1$

The resolvent of $-\Delta$ is singular at $z_0 = 0$ as the resolvent kernel is

$$R_0(x, y; z) = \frac{e^{i|x-y|\sqrt{z}}}{2i\sqrt{z}},$$

for $z \in \mathbb{C} \setminus \overline{\mathbb{R}_+}$, and where $\Im\sqrt{z} > 0$.

- There is a singularity at $z \to z_0 = 0$ which is an obstruction to a LAP in this framework.
- This singularity disappears under a well chosen perturbation (next two/three slides).

Here $\mathbf{X} = \mathcal{H} = L^2(\mathbb{R})$ and for $\mathbf{E}$ and $\mathbf{F}$ we consider spaces of the form

$$L^p_s(\mathbb{R}^d) = \left\{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d);\ \langle x\rangle^s u \in L^p(\mathbb{R}^d)\right\}$$

for $1 \le p \le \infty$ and $s \in \mathbb{R}$.

Schrödinger operators with decaying potentials in $L^2(\mathbb{R})$

Virtual levels and LAP estimates

Theorem

Let V be a measurable function on $\mathbb{R}$ such that $\langle x\rangle V \in L^1(\mathbb{R})$. There is the following dichotomy:

1. *Either*

$$R_V(z) = (-\Delta + V - zI)^{-1}, \qquad z \in \mathbb{C} \setminus \sigma(-\Delta + V)$$

   is uniformly bounded as a mapping $L^2_{s_1}(\mathbb{R}) \to L^2_{-s_2}(\mathbb{R})$ for some $s_1,\, s_2 > 1/2$, $s_1 + s_2 \ge 2$, for $z \in \mathbb{D}_\delta \setminus \overline{\mathbb{R}_+}$ with some $\delta > 0$.

2. *Or there is a nontrivial solution to*

$$(-\Delta + V)\Psi = 0, \qquad \Psi \in L^\infty(\mathbb{R}) \cap H^2_{\mathrm{loc}}(\mathbb{R}).$$

A remark when LAP holds

When the first alternative holds:

$$R_V(z) = (-\Delta + V - zI)^{-1}, \qquad z \in \mathbb{C} \setminus \sigma(-\Delta + V)$$

is uniformly bounded as a mapping $L^2_{s_1}(\mathbb{R}) \to L^2_{-s_2}(\mathbb{R})$ for some $s_1,\, s_2 > 1/2$, $s_1 + s_2 \geq 2$, for $z \in \mathbb{D}_\delta \setminus \overline{\mathbb{R}_+}$ with some $\delta > 0$.
Then $R_V(z)$ has a limit as $z \to z_0 = 0$, $z \in \mathbb{C} \setminus (-\Delta + V)$

1. In the strong operator topology of $\mathscr{B}\big(L^2_s(\mathbb{R}), L^2_{-s'}(\mathbb{R})\big)$ for all pairs of s, s' such that s, $s' > 1/2$, $s + s' \geq 2$ (in the uniform operator topology if $s + s' > 2$);
2. In the weak* operator topology of $\mathscr{B}\big(L^1_\varsigma(\mathbb{R}), L^\infty_{\varsigma-1}(\mathbb{R})\big)$, $0 \leq \varsigma \leq 1$.

A condition for the LAP

Lemma

Let V be a real-valued nonnegative locally L^2-integrable function on $\mathbb{R}$.

1. *If V is not identically zero, then there is no nontrivial solution to $(-\Delta + V)\Psi = 0$ such that $\Psi \in L^2_{-1}(\mathbb{R})$.*
2. *If, moreover, V is compactly supported and $s' \leq 3/2$, then there is no nontrivial solution to $(-\Delta + V)\Psi = 0$ such that $\Psi \in L^2_{-s'}(\mathbb{R})$.*

"General" Schrödinger operators in $L^2(\mathbb{R})$

virtual levels and LAP estimates

Theorem

- *Let $s_1, s_2 > 1/2$, $s_1 + s_2 \geq 2$, $s_2 \leq 3/2$,*
- *And assume that $W : L^2_{-s_2}(\mathbb{R}) \to L^2_{s_1}(\mathbb{R})$ is Δ-compact (with Δ considered in $L^2_{-s_2}(\mathbb{R})$).*

There is the following dichotomy:

1. *Either $R_W(z) = (-\Delta + W - zI)^{-1}$, $z \in \mathbb{C} \setminus \sigma(-\Delta + W)$ is bounded as a mapping $L^2_{s_1}(\mathbb{R}) \to L^2_{-s_2}(\mathbb{R})$ uniformly in $z \in \mathbb{D}_\delta \setminus \overline{\mathbb{R}_+}$ with some $\delta > 0$;*
2. *Or there is a nontrivial solution to $(-\Delta + W)\Psi = 0$, $\Psi \in L^2_{-s_2}(\mathbb{R})$.*